Extending the Erdős–Ko–Rado theorem

Hajime Tanaka

Graduate School of Information Sciences,
Tohoku University

IWONT 2012
Institut Teknologi Bandung
Bandung, Indonesia
July 27, 2012

The Erdős–Ko–Rado theorem (1961)

- Ω : a finite set with $|\Omega| = v \geqslant 2d$
- $X = \{x \subseteq \Omega : |x| = d\}$

Theorem (Erdős–Ko–Rado, 1961)

Let $v \geqslant (t+1)(d-t+1)$ and let $Y \subseteq X$ be a t-intersecting family, i.e., $|x \cap y| \geqslant t$ for all $x, y \in Y$. Then

$$|Y| \leqslant \binom{v-t}{d-t}.$$

If $v > (t+1)(d-t+1)$ and if $|Y| = \binom{v-t}{d-t}$ then

$$Y = \{x \in X : u \subseteq x\}$$

for some $u \subseteq \Omega$ with $|u| = t$.

Distance-regular graphs

- $\Gamma = (X, R)$: a finite connected simple graph with diameter d
- ∂ : the path-length distance function
- Define $A_0, A_1, \ldots, A_d \in \mathbb{R}^{X \times X}$ by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{otherwise} \end{cases}$$

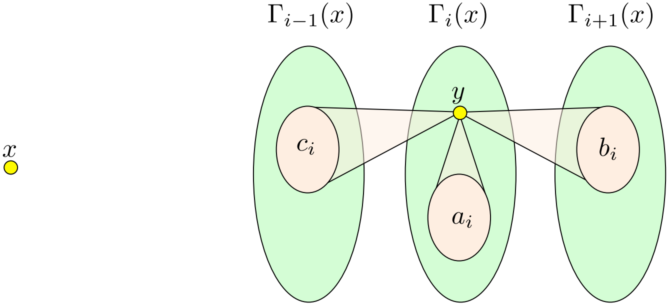
- Γ is distance-regular if there are integers a_i, b_i, c_i such that

$$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \ (i = 0, 1, \ldots, d)$$

where $A_{-1} = A_{d+1} = 0$.

$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \quad (i = 0, 1, \ldots, d)$

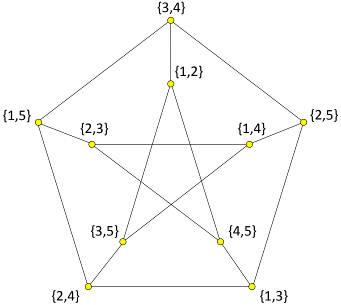
- For $x \in X$, set $\Gamma_i(x) = \{y \in X : \partial(x, y) = i\}$.



Johnson graphs

- Ω : a finite set with $|\Omega| = v \geqslant 2d$
- $X = \{x \subseteq \Omega : |x| = d\}$
- $x \sim_R y \Leftrightarrow |x \cap y| = d - 1 \quad (x, y \in X)$
- $\Gamma = J(v, d) = (X, R)$: the Johnson graph

- The complement of $J(5, 2)$ with $\Omega = \{1, 2, 3, 4, 5\}$:



$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \quad (i = 0, 1, \ldots, d)$

- $\Gamma = (X, R)$: a distance-regular graph with diameter d
- $A_0, A_1, \ldots, A_d$: the distance matrices of Γ
- $\theta_0, \theta_1, \ldots, \theta_d$: the distinct eigenvalues of A_1
- E_i : the orthogonal projection onto the eigenspace of A_1 with eigenvalue θ_i
- $\mathbb{R}[A_1] = \langle A_0, \ldots, A_d \rangle = \langle E_0, \ldots, E_d \rangle$: the Bose–Mesner algebra of Γ

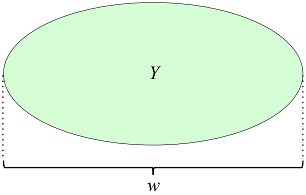
- For the rest of this talk, we suppose $\theta_0, \theta_1, \ldots, \theta_d$ is a Q-polynomial ordering.

Remark

For our examples of graphs, we have $\theta_0 > \theta_1 > \cdots > \theta_d$.

Width and dual width (Brouwer et al., 2003)

- $Y \subseteq X$: a nonempty subset of X
- $\chi \in \mathbb{R}^X$: the (column) characteristic vector of Y
- $w = \max\{i : \chi^{\mathsf{T}} A_i \chi \neq 0\}$: the width of Y
- $w^* = \max\{i : \chi^{\mathsf{T}} E_i \chi \neq 0\}$: the dual width of Y



Theorem (Brouwer–Godsil–Koolen–Martin, 2003)

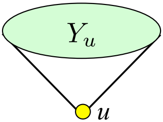
$$w + w^* \geqslant d.$$

Definition

We call Y a descendent of Γ if $w + w^* = d$.

The descendents of the Johnson graphs

- Ω : a finite set with $|\Omega| = v \geqslant 2d$
- $X = \{x \subseteq \Omega : |x| = d\}$
- $u \subseteq \Omega : |u| = i$
- $Y_u := \{x \in X : u \subseteq x\}$ satisfies $w = d - i$ and $w^* = i$:



Theorem (Brouwer et al., 2003)

Every descendent of $\Gamma = J(v, d)$ is isomorphic to some Y_u under the full automorphism group $\mathrm{Aut}(\Gamma)$ of Γ.

The Erdős–Ko–Rado theorem (1961)

- Ω : a finite set with $|\Omega| = v \geqslant 2d$
- $X = \{x \subseteq \Omega : |x| = d\}$

Theorem (Erdős–Ko–Rado, 1961)

Let $v \geqslant (t+1)(d-t+1)$ and let $Y \subseteq X$ be a t-intersecting family, i.e., $|x \cap y| \geqslant t$ for all $x, y \in Y$. Then

$$|Y| \leqslant \binom{v-t}{d-t}.$$

If $v > (t+1)(d-t+1)$ and if $|Y| = \binom{v-t}{d-t}$ then

$$Y = \{x \in X : u \subseteq x\}$$

for some $u \subseteq \Omega$ with $|u| = t$.

A “modern” treatment of the EKR theorem

- This is in fact a result about the Johnson graph $J(v, d)$.

Theorem (Erdős–Ko–Rado, 1961)

Let $v \geqslant (t+1)(d-t+1)$ *and let* $Y \subseteq X$ *be a* t*-intersecting family, i.e.,* $w \leqslant d-t$*. Then*

$$|Y| \leqslant \binom{v-t}{d-t}.$$

If $v > (t+1)(d-t+1)$ *and if* $|Y| = \binom{v-t}{d-t}$ *then*

$$Y = Y_u$$

for some $u \subseteq \Omega$ *with* $|u| = t$*.*

The LP bound (Delsarte, 1973)

- $\Gamma = (X, R)$: a distance-regular graph with diameter d
- $Y \subseteq X : w \leqslant d - t$ (i.e., "t-intersecting")
- $\chi \in \mathbb{R}^X$: the (column) characteristic vector of Y

- $M := \frac{1}{|Y|}\chi\chi^{\mathsf{T}} \in \mathbb{R}^{X \times X}$: nonnegative & positive semidefinite
- $\langle M, I \rangle = 1, \ \langle M, J \rangle = |Y|$ (where J is the all 1's matrix)
- $\langle M, A_i \rangle = \frac{1}{|Y|}\chi^{\mathsf{T}} A_i \chi = 0$ for $i = w+1, \ldots, d$

The LP bound (Delsarte, 1973), continued

- Consider the following SDP problem:

$$\ell_{\mathsf{LP}} = \max \langle M, J \rangle$$

subject to

1. $\langle M, I \rangle = 1$,
2. $\langle M, A_i \rangle = 0 \ \ (i = w+1, \ldots, d)$,
3. M : nonnegative & positive semidefinite.

- Then $|Y| \leqslant \ell_{\mathsf{LP}}$.

Remark

ℓ_{LP} is the strengthening of Lovász's ϑ-number due to Schrijver (1979).

- By projecting M to $\mathbb{R}[A_1]=\langle A_0,\ldots,A_d\rangle=\langle E_0,\ldots,E_d\rangle$, ℓ_{LP} turns to an LP:

$$(|Y|\leqslant)\ \ell_{\mathsf{LP}}=\max\langle M,J\rangle$$

subject to

1. $\langle M,I\rangle=1$,
2. $\langle M,A_i\rangle=0\ \ (i=w+1,\ldots,d)$,
3. $\sum_{i=0}^{d}\frac{\langle M,A_i\rangle}{\langle A_i,A_i\rangle}A_i=\sum_{i=0}^{d}\frac{\langle M,E_i\rangle}{\langle E_i,E_i\rangle}E_i\ \geqslant 0$ & $\succcurlyeq 0$, i.e.,
   $\langle M,A_i\rangle\geqslant 0\ \ (i=1,\ldots,w)$, $\langle M,E_i\rangle\geqslant 0\ \ (i=1,\ldots,d)$.

- A vector f (unique, if any) satisfying the following conditions gives a feasible solution to the dual problem:

(D1) $$f_0 = 1,\ f_1 = \cdots = f_t = 0,$$

(D2) $$f_{t+1} > 0, \ldots, f_d > 0,$$

(D3) $$(fQ^{\mathsf{T}})_1 = \cdots = (fQ^{\mathsf{T}})_{d-t} = 0,$$

where $|X|(E_0, E_1, \ldots, E_d) = (A_0, A_1, \ldots, A_d)Q$.

- By the duality of linear programming, we have

$$|Y| \leqslant (fQ^{\mathsf{T}})_0$$

and equality holds if and only if Y is a descendent of Γ.

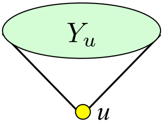
Remark

- There is a universal description of the vector f (T.).

Grassmann graphs

- V : a vector space over $\mathbb{F}_q$ with $\dim V = v \geqslant 2d$
- $X = \{x \leqslant V : \dim x = d\}$
- $x \sim_R y \Leftrightarrow \dim x \cap y = d - 1 \quad (x, y \in X)$
- $\Gamma = J_q(v, d) = (X, R)$: the Grassmann graph

- $u \leqslant V$: $\dim u = i$
- $Y_u := \{x \in X : u \leqslant x\}$ satisfies $w = d - i$ and $w^* = i$:



Theorem (T., 2006)

Every descendent of $\Gamma = J_q(v, d)$ is isomorphic to some Y_u under the full automorphism group $\mathrm{Aut}(\Gamma)$ of Γ.

The EKR theorem for Grassmann graphs

Theorem (T., 2006)

Let Y be a nonempty subset of $J_q(v,d)$ with width $w \leqslant d-t$, where $0 < t < d$. Then $|Y| \leqslant \begin{bmatrix} v-t \\ d-t \end{bmatrix}_q$, and equality holds if and only if Y is a descendent with $w = d-t$.

Remark

Partial results were previously obtained by Hsieh (1975), Frankl–Wilson (1986), Fu (1999).

Twisted Grassmann graphs

- V : a vector space over $\mathbb{F}_q$ with $\dim V = 2d+1$
- H : a hyperplane of V
- $X_1 := \{x \leqslant V : \dim x = d+1,\ x \not\leqslant H\}$
- $X_2 := \{x \leqslant H : \dim x = d-1\}$
- $X := X_1 \cup X_2$
- $x \sim_R y \iff 2\dim x \cap y = \dim x + \dim y - 2 \quad (x, y \in X)$
- $\Gamma = \tilde{J}_q(2d+1, d) = (X, R)$: the twisted Grassmann graph

Remark

- $\tilde{J}_q(2d+1, d)$ was constructed by Van Dam and Koolen (2005).
- $\tilde{J}_q(2d+1, d)$ has the same parameters as $J_q(2d+1, d)$.
- X_1, X_2 are the orbits of $\mathrm{Aut}(\tilde{J}_q(2d+1, d))$ on X.
- The induced subgraph on X_2 is $J_q(2d, d-1)$.

- $u \leqslant H : \dim u = i - 1$
- $Y_u := \{x \in X_2 : u \leqslant x\}$ satisfies $w = d - i$ and $w^* = i$.

Theorem (T., 2011)

Every descendent of $\Gamma = \tilde{J}_q(2d+1, d)$ with $0 < w < d$ is of the form Y_u.

Using the same vector f for $J_q(2d+1, d)$, we get:

Theorem (T.)

Let Y be a nonempty subset of $\tilde{J}_q(2d+1, d)$ with width $w \leqslant d - t$, where $0 < t < d$. Then $|Y| \leqslant \begin{bmatrix} 2d+1-t \\ d-t \end{bmatrix}_q$, and equality holds if and only if $Y = Y_u$ for some $u \leqslant H$ with $\dim u = t - 1$.