Extending the Erdős–Ko–Rado theorem

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The Erdős–Ko–Rado theorem (1961)

- Ω : a finite set with $|\Omega| = v \geqslant 2d$
- $X = \{x \subseteq \Omega : |x| = d\}$

Theorem (Erdős–Ko–Rado, 1961; Wilson, 1984)

Let $v \geqslant (t+1)(d-t+1)$ and let $Y \subseteq X$ be a t-intersecting family, i.e., $|x \cap y| \geqslant t$ for all $x, y \in Y$. Then

$$|Y| \leqslant \binom{v-t}{d-t}.$$

If $v > (t+1)(d-t+1)$ and if $|Y| = \binom{v-t}{d-t}$ then

$$Y = \{x \in X : u \subseteq x\}$$

for some $u \subseteq \Omega$ with $|u| = t$.

Contents

1. a unified and “modern” approach to the Erdős–Ko–Rado theorem for Q-polynomial distance-regular graphs
2. an application to the twisted Grassmann graphs discovered in 2005 by Van Dam and Koolen

Reference

- H. Tanaka, The Erdős–Ko–Rado theorem for twisted Grassmann graphs, to appear in Combinatorica; arXiv:1012.5692.
- H. Tanaka, The Erdős–Ko–Rado basis for a Leonard system, in preparation.

Distance-regular graphs

- $\Gamma = (X, R)$: a finite connected simple graph with diameter d
- ∂ : the path-length distance function
- Define $A_0, A_1, \ldots, A_d \in \mathbb{R}^{X \times X}$ by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i \\ 0 & \text{otherwise} \end{cases}$$

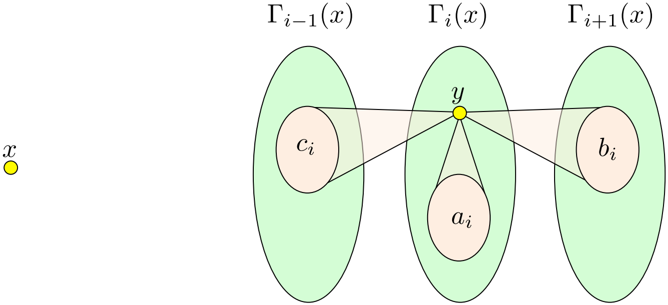
- Γ is distance-regular if there are integers a_i, b_i, c_i such that

$$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \ (i = 0, 1, \ldots, d)$$

where $A_{-1} = A_{d+1} = 0$.

$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \quad (i = 0, 1, \ldots, d)$

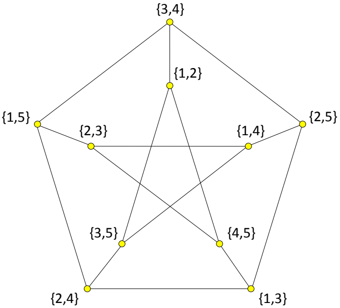
- For $x \in X$, set $\Gamma_i(x) = \{y \in X : \partial(x, y) = i\}$.



Johnson graphs

- Ω : a finite set with $|\Omega| = v \geqslant 2d$
- $X = \{x \subseteq \Omega : |x| = d\}$
- $x \sim_R y \iff |x \cap y| = d - 1 \quad (x, y \in X)$
- $\Gamma = J(v, d) = (X, R)$: the Johnson graph

- The complement of $J(5, 2)$ with $\Omega = \{1, 2, 3, 4, 5\}$:



$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \quad (i = 0, 1, \ldots, d)$

- $\Gamma = (X, R)$: a distance-regular graph with diameter d
- $A_0, A_1, \ldots, A_d$: the distance matrices of Γ
- $\theta_0, \theta_1, \ldots, \theta_d$: the distinct eigenvalues of A_1
- E_i : the orthogonal projection onto the eigenspace of A_1 with eigenvalue θ_i
- $\mathbb{R}[A_1] = \langle A_0, \ldots, A_d \rangle = \langle E_0, \ldots, E_d \rangle$: the Bose–Mesner algebra of Γ

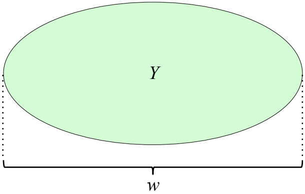
- For the rest of this talk, we suppose $\theta_0, \theta_1, \ldots, \theta_d$ is a Q-polynomial ordering.

Remark

For our examples of graphs, we have $\theta_0 > \theta_1 > \cdots > \theta_d$.

Width and dual width (Brouwer et al., 2003)

- $Y \subseteq X$: a nonempty subset of X
- $\chi \in \mathbb{R}^X$: the (column) characteristic vector of Y
- $w = \max\{i : \chi^{\mathsf{T}} A_i \chi \neq 0\}$: the width of Y
- $w^* = \max\{i : \chi^{\mathsf{T}} E_i \chi \neq 0\}$: the dual width of Y



Theorem (Brouwer–Godsil–Koolen–Martin, 2003)

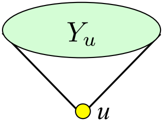
$$w + w^* \geqslant d.$$

Definition

We call Y a descendent of Γ if $w + w^* = d$.

The descendents of the Johnson graphs

- Ω : a finite set with $|\Omega| = v \geqslant 2d$
- $X = \{x \subseteq \Omega : |x| = d\}$
- $u \subseteq \Omega : |u| = i$
- $Y_u := \{x \in X : u \subseteq x\}$ satisfies $w = d - i$ and $w^* = i$:



Theorem (Brouwer et al., 2003)

Every descendent of $\Gamma = J(v, d)$ is isomorphic to some Y_u under the full automorphism group $\mathrm{Aut}(\Gamma)$ of Γ.

The Erdős–Ko–Rado theorem (1961)

- Ω : a finite set with $|\Omega| = v \geqslant 2d$
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Theorem (Erdős–Ko–Rado, 1961; Wilson, 1984)

Let $v \geqslant (t+1)(d-t+1)$ and let $Y \subseteq X$ be a t-intersecting family, i.e., $|x \cap y| \geqslant t$ for all $x, y \in Y$. Then

$$|Y| \leqslant \binom{v-t}{d-t}.$$

If $v > (t+1)(d-t+1)$ and if $|Y| = \binom{v-t}{d-t}$ then

$$Y = \{x \in X : u \subseteq x\}$$

for some $u \subseteq \Omega$ with $|u| = t$.

A “modern” treatment of the EKR theorem

- This is in fact a result about the Johnson graph $J(v,d)$.

Theorem (Erdős–Ko–Rado, 1961; Wilson, 1984)

Let $v \geqslant (t+1)(d-t+1)$ and let $Y \subseteq X$ be a t-intersecting family, i.e., $w \leqslant d-t$. Then

$$|Y| \leqslant \binom{v-t}{d-t}.$$

If $v > (t+1)(d-t+1)$ and if $|Y| = \binom{v-t}{d-t}$ then

$$Y = Y_u$$

for some $u \subseteq \Omega$ with $|u| = t$.

The LP bound (Delsarte, 1973)

- $\Gamma = (X, R)$: a distance-regular graph with diameter d
- $Y \subseteq X$: $w \leqslant d - t$ (i.e., "t-intersecting")
- $\chi \in \mathbb{R}^X$: the (column) characteristic vector of Y

- $M := \frac{1}{|Y|}\chi\chi^{\mathsf{T}} \in \mathbb{R}^{X \times X}$: nonnegative & positive semidefinite
- $\langle M, I \rangle = 1, \ \langle M, J \rangle = |Y|$ (where J is the all 1's matrix)
- $\langle M, A_i \rangle = \frac{1}{|Y|}\chi^{\mathsf{T}} A_i \chi = 0$ for $i = w+1, \ldots, d$

The LP bound (Delsarte, 1973), continued

- Consider the following SDP problem:

$$\ell_{\mathrm{LP}} = \max\langle M, J\rangle$$

subject to

1. $\langle M, I\rangle = 1$,
2. $\langle M, A_i\rangle = 0 \quad (i = w+1, \ldots, d)$,
3. M : nonnegative & positive semidefinite.

- Then $|Y| \leqslant \ell_{\mathrm{LP}}$.

Remark

ℓ_{LP} is the strengthening of Lovász's ϑ-number due to Schrijver (1979).

- By projecting M to $\mathbb{R}[A_1] = \langle A_0,\ldots,A_d\rangle = \langle E_0,\ldots,E_d\rangle$, ℓ_{LP} turns to an LP:

$$(|Y| \leqslant)\ \ell_{\text{LP}} = \max\langle M,J\rangle$$

subject to

1. $\langle M,I\rangle = 1$,
2. $\langle M,A_i\rangle = 0 \ \ (i = w+1,\ldots,d)$,
3. $\sum_{i=0}^{d} \frac{\langle M,A_i\rangle}{\langle A_i,A_i\rangle} A_i = \sum_{i=0}^{d} \frac{\langle M,E_i\rangle}{\langle E_i,E_i\rangle} E_i \ \geqslant 0$ & $\succcurlyeq 0$, i.e.,
   $\langle M,A_i\rangle \geqslant 0 \ \ (i = 1,\ldots,w)$, $\langle M,E_i\rangle \geqslant 0 \ \ (i = 1,\ldots,d)$.

- A vector $\boldsymbol{f} = (f_0, f_1, \ldots, f_d)$ (unique, if any) satisfying the following conditions gives a feasible solution to the dual problem:

(D1) $$f_0 = 1,\ f_1 = \cdots = f_t = 0,$$

(D2) $$f_{t+1} > 0, \ldots, f_d > 0,$$

(D3) $$(\boldsymbol{f}Q^{\mathsf{T}})_1 = \cdots = (\boldsymbol{f}Q^{\mathsf{T}})_{d-t} = 0,$$

where $|X|(E_0, E_1, \ldots, E_d) = (A_0, A_1, \ldots, A_d)Q$.

- By the duality of LP, we have

$$|Y| \leqslant (\boldsymbol{f}Q^{\mathsf{T}})_0$$

and equality holds if and only if Y is a descendent of Γ.

$f_0 = 1,\ f_1 = \cdots = f_t = 0,\ (fQ^\mathsf{T})_1 = \cdots = (fQ^\mathsf{T})_{d-t} = 0$

- Under certain conditions, the vector satisfying (D1)–(D3) was constructed in each of the following cases:

Γ	f	$(fQ^\mathsf{T})_0$
Johnson $J(v,d)$	Wilson (1984)	$\binom{v-t}{d-t}$
Hamming $H(d,q)$	MDS weight enumerators	q^{d-t}
Grassmann $J_q(v,d)$	Frankl–Wilson (1986)	$\begin{bmatrix} v-t \\ d-t \end{bmatrix}_q$
bilinear forms $\mathrm{Bil}_q(d,e)$	(d,e,t,q)-Singleton systems, Delsarte (1978)	$q^{(d-t)e}$

$|X|(E_0, E_1, \ldots, E_d) = (A_0, A_1, \ldots, A_d)Q$

Theorem (Leonard, 1982; Bannai–Ito, 1984; Terwilliger, 2001)

Γ *is associated with the* *parameter array*

$$p(\Gamma) = \left(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d\right)$$

such that

$$(*) \qquad \frac{Q_{ij}}{Q_{0j}} = \sum_{h=0}^{j} \frac{(\theta_i - \theta_0)\ldots(\theta_i - \theta_{h-1})(\theta_j^* - \theta_0^*)\ldots(\theta_j^* - \theta_{h-1}^*)}{\varphi_1 \ldots \varphi_h}$$

where

$$Q_{0j} = \frac{\varphi_1 \ldots \varphi_j}{\phi_{d-j+1} \ldots \phi_d} \frac{(\theta_0 - \theta_1)\ldots(\theta_0 - \theta_d)}{(\theta_i - \theta_0)\ldots(\theta_i - \theta_{i-1})(\theta_i - \theta_{i+1})\ldots(\theta_i - \theta_d)}$$

Remark

The RHS of (*) is a value of a q-Racah polynomial or its certain limiting/special case (in the Askey & Bannai–Ito schemes).

$f_0 = 1,\ f_1 = \cdots = f_t = 0,\ (fQ^{\mathsf{T}})_1 = \cdots = (fQ^{\mathsf{T}})_{d-t} = 0$

Theorem (T.)

$$f_j = \frac{\eta_{d-t}(\theta_0)}{\eta_d(\theta_0)\eta^*_t(\theta^*_0)} \frac{\phi_{d-j+1}\dots\phi_d}{\varphi_2\dots\varphi_j(\theta_j-\theta_0)} \left(\sum_{\ell=t+1}^{j} \frac{\tau_\ell(\theta_j)\eta^*_{\ell-1}(\theta^*_0)\vartheta_\ell}{\phi_{d-\ell+1}\dots\phi_{d-t}} \right)$$

for $j = t+1, \dots, d$, *where*

$$\tau_i(z) = (z-\theta_0)\dots(z-\theta_{i-1}), \quad \eta_i(z) = (z-\theta_d)\dots(z-\theta_{d-i+1}),$$
$$\tau^*_i(z) = (z-\theta^*_0)\dots(z-\theta^*_{i-1}), \quad \eta^*_i(z) = (z-\theta^*_d)\dots(z-\theta^*_{d-i+1}),$$
$$\vartheta_i = \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d}.$$

Generic case; q-Racah

The most general form of the parameter array is as follows:

$p(\Gamma) = p(q, r_1, r_2, s, s^*, d)$ where $r_1 r_2 = s s^* q^{d+1} \neq 0$,

$$\theta_i = \theta_0 + h(1-q^i)(1-sq^{i+1})q^{-i},$$
$$\theta^*_i = \theta^*_0 + h^*(1-q^i)(1-s^*q^{i+1})q^{-i}$$

for $i = 0, \ldots, d$,

$$\varphi_i = hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1})(1-r_1q^i)(1-r_2q^i),$$
$$\phi_i = hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1})(r_1-s^*q^i)(r_2-s^*q^i)/s^*$$

for $i = 1, \ldots, d$.

Generic case; q-Racah

Example

$$f_j = \frac{s^{*j-1} q^{(d+1)(j-1)+t} (q^{1-j};q)_t (sq^{j+2};q)_t (sq/r_1;q)_j (sq/r_2;q)_j}{(1-sq^{t+1}/r_1)(1-sq^{t+1}/r_2)(q;q)_t (sq^2;q)_t (r_1 q^2;q)_{j-1} (r_2 q^2;q)_{j-1}} \times {}_4\phi_3\left(\begin{matrix} q^{t-j+1}, sq^{t+j+2}, q^{t-d-1}/s^*, q \\ q^{t+1}, sq^{t+2}/r_1, sq^{t+2}/r_2 \end{matrix} \middle| q;q\right)$$

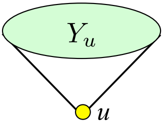
for $j = t+1, \ldots, d$, and

$$(fQ^{\mathsf{T}})_0 = \frac{(sq^{t+2};q)_{d-t} (s^* q^2;q)_{d-t}}{r_1^{d-t} q^{d-t} (sq^{t+1}/r_1;q)_{d-t} (s^* q/r_1;q)_{d-t}}.$$

Grassmann graphs

- V : a vector space over $\mathbb{F}_q$ with $\dim V = v \geqslant 2d$
- $X = \{x \leqslant V : \dim x = d\}$
- $x \sim_R y \iff \dim x \cap y = d - 1 \quad (x, y \in X)$
- $\Gamma = J_q(v, d) = (X, R)$: the Grassmann graph

- $u \leqslant V$: $\dim u = i$
- $Y_u := \{x \in X : u \leqslant x\}$ satisfies $w = d - i$ and $w^* = i$:



Theorem (T., 2006)

Every descendent of $\Gamma = J_q(v, d)$ is isomorphic to some Y_u under the full automorphism group $\mathrm{Aut}(\Gamma)$ of Γ.

The EKR theorem for Grassmann graphs

Theorem (T., 2006)

Let Y be a nonempty subset of $J_q(v,d)$ with width $w \leqslant d-t$, where $0 < t < d$. Then $|Y| \leqslant \begin{bmatrix} v-t \\ d-t \end{bmatrix}_q$, and equality holds if and only if Y is a descendent with $w = d-t$.

Remark

Partial results were previously obtained by Hsieh (1975), Frankl–Wilson (1986), Fu (1999).

Twisted Grassmann graphs

- V : a vector space over $\mathbb{F}_q$ with $\dim V = 2d+1$
- H : a hyperplane of V
- $X_1 := \{x \leqslant V : \dim x = d+1,\ x \nleqslant H\}$
- $X_2 := \{x \leqslant H : \dim x = d-1\}$
- $X := X_1 \cup X_2$
- $x \sim_R y \iff 2\dim x \cap y = \dim x + \dim y - 2 \quad (x, y \in X)$
- $\Gamma = \tilde{J}_q(2d+1, d) = (X, R)$: the twisted Grassmann graph

Remark

- $\tilde{J}_q(2d+1, d)$ was constructed by Van Dam and Koolen (2005).
- $\tilde{J}_q(2d+1, d)$ has the same parameters as $J_q(2d+1, d)$.
- X_1, X_2 are the orbits of $\mathrm{Aut}(\tilde{J}_q(2d+1, d))$ on X.
- The induced subgraph on X_2 is $J_q(2d, d-1)$.

$X_1 = \{x \leqslant V : \dim x = d+1, x \not\leqslant H\}$, $X_2 = \{x \leqslant H : \dim x = d-1\}$

- $u \leqslant H : \dim u = i - 1$
- $Y_u := \{x \in X_2 : u \leqslant x\}$ satisfies $w = d - i$ and $w^* = i$.

Theorem (T., 2011)

Every descendent of $\Gamma = \tilde{J}_q(2d+1, d)$ with $0 < w < d$ is of the form Y_u.

Using the same vector f for $J_q(2d+1, d)$, we get:

Theorem (T.)

Let Y be a nonempty subset of $\tilde{J}_q(2d+1, d)$ with width $w \leqslant d - t$, where $0 < t < d$. Then $|Y| \leqslant \begin{bmatrix} 2d+1-t \\ d-t \end{bmatrix}_q$, and equality holds if and only if $Y = Y_u$ for some $u \leqslant H$ with $\dim u = t - 1$.