# Spatial search on Johnson graphs by continuous-time quantum walk

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# How I started this work

- I am a mathematician specializing in **distance-regular graphs** and **association schemes**, in particular their **Terwilliger algebras**.
- Etsuo Segawa (a specialist in quantum walks) invited Portugal to Tohoku U in Jan–Feb 2018, and Portugal gave a series of lectures on search algorithms based on quantum walks.
- Sabri was a student of Segawa, and became my Ph.D. student in Sep 2019.

$\Longrightarrow$ I started studying quantum walks.

# General set-up

vertex set

edge set

- $G = (V, E)$ : a finite simple graph, where $|V| = N$
- $\mathscr{H} = \mathrm{span}\{ |v\rangle : v \in V\}$, where $\langle v | v' \rangle = \delta_{v,v'}$
- $A$ : the **adjacency matrix** of $G$ :

$$A_{v,v'} = \begin{cases} 1 & \text{if } v \sim v' \\ 0 & \text{otherwise} \end{cases} \qquad (v, v' \in V)$$

- $w \in V$ : the marked vertex
- Consider a Hamiltonian of the form (Childs–Goldstone, 2004)

$$H = -\gamma A - |w\rangle\langle w|$$

where $\gamma \in \mathbb{R}$.

# General set-up

$H = -\gamma A - |w\rangle\langle w|$

- $|\psi(0)\rangle \in \mathscr{H}$ : the initial state
- $|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht}|\psi(0)\rangle$ : the state at time $t$
- $|\langle w|\psi(t)\rangle|^2$ : the finding probability at time $t$

**Problem.**

Choose $|\psi(0)\rangle \in \mathscr{H}$, $\gamma \in \mathbb{R}$, $t \in \mathbb{R}$ which maximize the finding probability !!

$O(\sqrt{N})$

**Remark.**

- $A \in$ the adjacency algebra
- $H, \mathrm{e}^{-\mathrm{i}Ht} \in$ the Terwilliger algebra w.r.t. $w$

# Some previous work

- complete graphs (Childs–Goldstone, 2004) ← DRGs
- hypercubes (Childs–Goldstone, 2004) ← DRGs
- Cartesian powers of cycles (Childs–Goldstone, 2004)
- strongly regular graphs (Janmark–Meyer–Wong, 2014) ← DRGs
- Johnson graphs with diameter $3$ (Wong, 2016) ← DRGs
- Erdős–Renyi graphs (Chakraborty–Novo–Ambainis–Omar, 2016)

# Today's topic

- Johnson graphs with any fixed diameter

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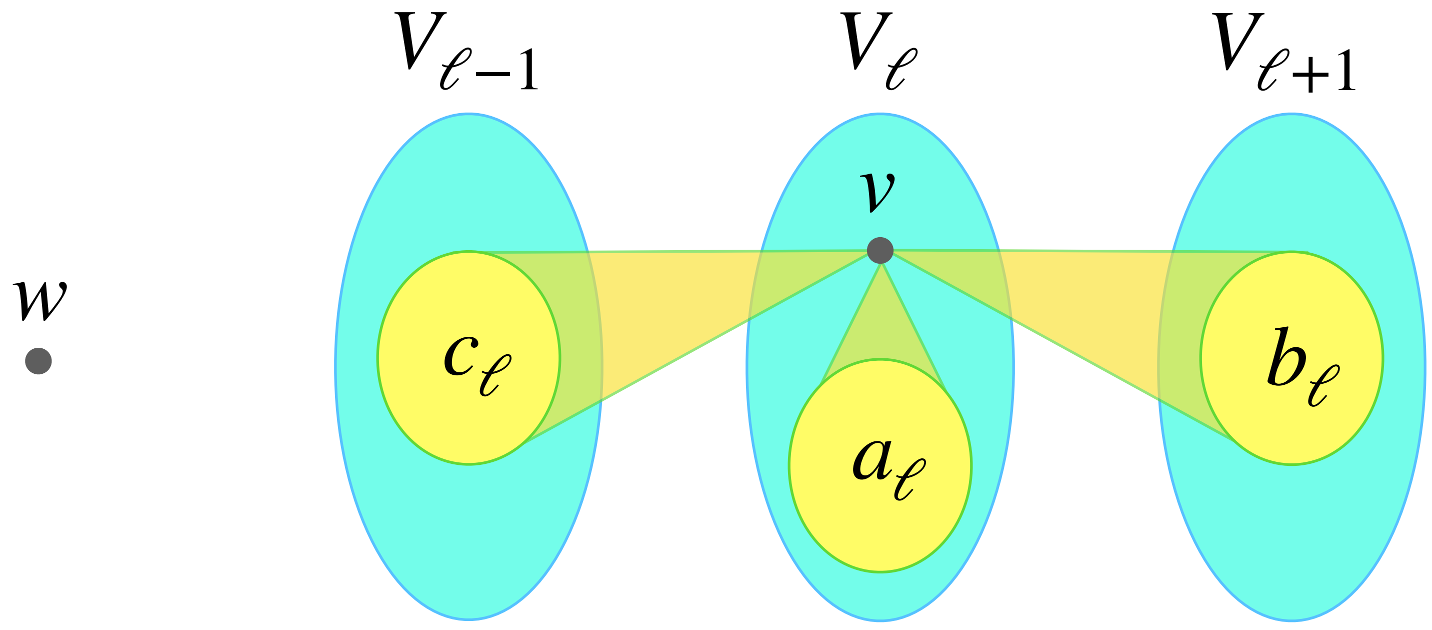
# Johnson graphs

- $n, k \in \mathbb{N} \ \ (1 \leqslant k \leqslant n/2)$
- $V$ : the set of $k$-subsets of $\{1,2,\ldots,n\} \implies N = \binom{n}{k}$
- $v \sim v' \overset{\text{def}}{\iff} |v \cap v'| = k-1 \quad (v, v' \in V)$
- $G = J(n,k)$

- $V_\ell = \{v \in V : |v \cap w| = k - \ell\} \ \ (0 \leqslant \ell \leqslant k)$



$a_\ell = \ell(n-2\ell)$

$b_\ell = (k-\ell)(n-k-\ell)$

$c_\ell = \ell^2$

$V_\ell = \{v : |v \cap w| = k - \ell\}$

- $|V_\ell\rangle = \dfrac{1}{\sqrt{|V_\ell|}} \sum_{v \in V_\ell} |v\rangle \quad (0 \leqslant \ell \leqslant k)$
- $|V_0\rangle = |w\rangle$
- $\mathscr{H}_{\mathrm{inv}} = \mathrm{span}\{|V_\ell\rangle : 0 \leqslant \ell \leqslant k\}$
- $\mathscr{H}_{\mathrm{inv}}$ : invariant under $A, |w\rangle\langle w|, H, \mathrm{e}^{-\mathrm{i}Ht}$

**We will work with $\mathscr{H}_{\mathrm{inv}}$ !!**

**Remark.**

- $\mathscr{H}_{\mathrm{inv}}$ : the **primary module** for the Terwilliger algebra

# General strategy

$H = -\gamma A - |w\rangle\langle w|$

- $|\psi(0)\rangle := |s\rangle = \dfrac{1}{\sqrt{N}} \sum_{v \in V} |v\rangle \in \mathscr{H}_{\mathrm{inv}}$
- Find eigenvalues $\rho_1, \rho_2$ and eigenvectors $|\xi_1\rangle, |\xi_2\rangle$ of $H$

  s.t. $|s\rangle \approx \dfrac{1}{2}|\xi_1\rangle - \dfrac{1}{2}|\xi_2\rangle, \quad |w\rangle \approx \dfrac{1}{2}|\xi_1\rangle + \dfrac{1}{2}|\xi_2\rangle.$
- Take the smallest $t > 0$ s.t. $(\rho_1 - \rho_2)\, t \equiv \pi \pmod{2\pi}$.

$$\Longrightarrow \quad |\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht}|s\rangle \approx \frac{\mathrm{e}^{\mathrm{i}\rho_1 t}}{2}|\xi_1\rangle - \frac{\mathrm{e}^{\mathrm{i}\rho_2 t}}{2}|\xi_2\rangle$$

$$= \mathrm{e}^{\mathrm{i}\rho_1 t}\left(\frac{1}{2}|\xi_1\rangle - \frac{\mathrm{e}^{\mathrm{i}\pi}}{2}|\xi_2\rangle\right) \approx \mathrm{e}^{\mathrm{i}\rho_1 t}|w\rangle$$

$\therefore \ |\langle w|\psi(t)\rangle|^2 \approx 1$ !!

# Johnson graphs with fixed diameter

**Fix $k$ and let $n \to \infty$ !!**

- The matrix expression of $A$ w.r.t. $|V_0\rangle, |V_1\rangle, \ldots, |V_k\rangle$ is

$$D \cdot \begin{pmatrix} a_0 & b_0 & & & \\ c_1 & a_1 & b_1 & & \\ & c_2 & \cdot & \cdot & \\ & & \cdot & a_{k-1} & b_{k-1} \\ & & & c_k & a_k \end{pmatrix} \cdot D^{-1}$$

where $D = \mathrm{diag}(|V_0|, \ldots, |V_k|)$.

difficult to handle !!

# Johnson graphs with fixed diameter

- $A$ has $k+1$ distinct eigenvalues

decreasing order

$$\theta_\ell = (k-\ell)(n-k-\ell)-\ell \quad (0 \leqslant \ell \leqslant k).$$

- $E_\ell$ : the projection onto the eigenspace in $\mathscr{H}$ w.r.t. $\theta_\ell$
- $|\theta_\ell\rangle = \dfrac{1}{\| E_\ell |w\rangle \|} E_\ell |w\rangle \in \mathscr{H}_{\mathrm{inv}} \quad (0 \leqslant \ell \leqslant k)$
- $\mathscr{H}_{\mathrm{inv}} = \mathrm{span}\{ |\theta_\ell\rangle : 0 \leqslant \ell \leqslant k\}$
- $\| E_\ell |w\rangle \|^2 = \dfrac{\binom{n}{\ell} - \binom{n}{\ell-1}}{\binom{n}{k}} = \dfrac{k!(n-k)!(n-2\ell+1)}{\ell!(n-\ell+1)!}$

$$\mathscr{H}_{\text{inv}} = \operatorname{span}\{\,|\theta_\ell\rangle : 0 \leqslant \ell \leqslant k\}$$

- ~~The matrix expression of $A$ w.r.t. $|\theta_0\rangle, |\theta_1\rangle, \ldots, |\theta_k\rangle$ is~~

$$A = \operatorname{diag}(\theta_0, \ldots, \theta_k).$$

- ~~Easy to see that the matrix expression of $|w\rangle\langle w|$ is~~

$$|w\rangle\langle w| = \Big(\|E_\ell\,|w\rangle\|\,\|E_{\ell'}\,|w\rangle\|\Big)_{\ell,\ell'=0}^{k}.$$

- $H = -\gamma\operatorname{diag}(\theta_0, \ldots, \theta_k) - \Big(\|E_\ell\,|w\rangle\|\,\|E_{\ell'}\,|w\rangle\|\Big)_{\ell,\ell'=0}^{k}.$

- For convergence, however, we will work with

$$-\eta H, \quad \text{where} \quad \eta := \frac{1}{\gamma n}.$$

$$-\eta H,\ \eta = 1/\gamma n$$

viewed as a variable

- $\epsilon := \dfrac{1}{\sqrt{n}} \longleftarrow$ viewed as a variable

- $$\begin{aligned} -\eta H &= \eta\big(\gamma A + |w\rangle\langle w|\big) \\ &= \epsilon^2 A + \eta |w\rangle\langle w| \\ &= \mathrm{diag}(r_0(\epsilon), r_1(\epsilon), \ldots, r_k(\epsilon)) + \eta\big(p_\ell(\epsilon)p_{\ell'}(\epsilon)\big)_{\ell,\ell'=0}^{k} \end{aligned}$$

where

$$r_\ell(\epsilon) := \epsilon^2\theta_\ell = (k-\ell)(1-(k+\ell)\epsilon^2) - \ell\epsilon^2$$

$$\begin{aligned} p_\ell(\epsilon) &:= \| E_\ell |w\rangle \| \\ &= \epsilon^{k-\ell}\sqrt{\frac{k!(1-(2\ell-1)\epsilon^2)}{\ell!(1-(\ell-1)\epsilon^2)\cdots(1-(k-1)\epsilon^2)}} \end{aligned}$$

- We are now concerned with eigenvalues and eigenvectors of

$$-\eta H = \mathrm{diag}(r_0(\epsilon), r_1(\epsilon), \ldots, r_k(\epsilon)) + \eta\big(p_\ell(\epsilon)p_{\ell'}(\epsilon)\big)_{\ell,\ell'=0}^{k}$$

variables

that "connect"

$$|s\rangle = |\theta_0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad |w\rangle = \begin{pmatrix} p_0(\epsilon) \\ p_1(\epsilon) \\ \vdots \\ p_k(\epsilon) \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad (\epsilon \to 0).$$

- For $(\epsilon, \eta) = (0,k)$, we have

$$-\eta H = \mathrm{diag}(k, k-1,\ldots,2,1,k).$$

**Proposition.**

- For every $a \in \mathbb{R}\backslash\{0\}$, there exist analytic functions

$$\eta(\epsilon),\ \lambda(\epsilon),\ \xi_\ell(\epsilon)\ (0 \leqslant \ell \leqslant k)$$

s.t. $\eta(0) = k,\ \xi_0(\epsilon) \equiv a,$ and

$$-\eta(\epsilon) H \,|\, \xi(\epsilon)\rangle = \lambda(\epsilon) \,|\, \xi(\epsilon)\rangle,$$

where

$$|\,\xi(\epsilon)\rangle = \begin{pmatrix} \xi_0(\epsilon) \\ \vdots \\ \xi_k(\epsilon) \end{pmatrix} \longrightarrow \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \ (\epsilon \to 0).$$

**Proposition** (continued)**.**

- Moreover,

$$\lambda(\epsilon) = r_0(\epsilon) + \frac{\sqrt{k!}\,k}{a}\epsilon^k + O(\epsilon^{k+1}),$$

$$\eta(\epsilon) = \left(\sum_{\ell=1}^{k} \frac{p_\ell(\epsilon)^2}{r_0(\epsilon) - r_\ell(\epsilon)}\right)^{-1} + \sqrt{k!}\,k\left(\frac{1}{a} - a\right)\epsilon^k + O(\epsilon^{k+1})\,.$$

# Perturbation theory

- $X(\epsilon) = (X_{i,j}(\epsilon))_{i,j=1}^{d}$ : an analytic $d \times d$ matrix function

  **Example :** $X(\epsilon) = X_0 + \epsilon X_1$

- Assume that eigenvalues and eigenvectors are analytic functions of $\epsilon$ :

$$X(\epsilon)\,|\,\xi(\epsilon)\rangle = \lambda(\epsilon)\,|\,\xi(\epsilon)\rangle$$

- Then we can derive many identities by equating coefficients of $\epsilon^m$ $(m = 0,1,2,\ldots)$ on both sides.

**Theorem** (Implicit Function Theorem)**.**

- $\varphi_i(\boldsymbol{x}, \boldsymbol{y})\ \ (1 \leqslant i \leqslant q)$ : analytic functions $(\boldsymbol{x} \in \mathbb{C}^p, \boldsymbol{y} \in \mathbb{C}^q)$
- $\varphi_i(\boldsymbol{x}^0, \boldsymbol{y}^0) = 0\ \ (1 \leqslant i \leqslant q)$
- $\det\left(\dfrac{\partial \varphi_j}{\partial y_i}\right)_{i,j=1}^{q}\Bigg|_{(\boldsymbol{x}^0,\boldsymbol{y}^0)} \neq 0$

$\Longrightarrow$ ① $\exists\, U$ : a neighborhood of $(\boldsymbol{x}^0, \boldsymbol{y}^0)$

② $g_1(\boldsymbol{x}), \ldots, g_q(\boldsymbol{x})$ : analytic functions

s.t., on $U$,

$$\varphi_i(\boldsymbol{x}, \boldsymbol{y}) = 0\ (1 \leqslant i \leqslant q) \iff \boldsymbol{y} = (g_1(\boldsymbol{x}), \ldots, g_q(\boldsymbol{x}))$$

determined by $a \in \mathbb{R}\setminus\{0\}$

- Recall

$$\eta(\epsilon) = \left(\sum_{\ell=1}^{k} \frac{p_\ell(\epsilon)^2}{r_0(\epsilon) - r_\ell(\epsilon)}\right)^{-1} + \sqrt{k!}\,k\left(\frac{1}{a} - a\right)\epsilon^k + O(\epsilon^{k+1})\,.$$

- For the search algorithm, we choose $\eta(\epsilon) = \eta^\circ(\epsilon)$, where

$$\eta^\circ(\epsilon) = \left(\sum_{\ell=1}^{k} \frac{p_\ell(\epsilon)^2}{r_0(\epsilon) - r_\ell(\epsilon)}\right)^{-1}.$$

- Then we have two “approximate” eigenvectors

$$\underset{a=1}{|\,\xi^+(\epsilon)\rangle} \longrightarrow \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad \underset{a=-1}{|\,\xi^-(\epsilon)\rangle} \longrightarrow \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad (\epsilon \to 0).$$

$$|s\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad |w\rangle \approx \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad |\xi^+\rangle \approx \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad |\xi^-\rangle \approx \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore \quad |s\rangle \approx \frac{1}{2}|\xi_1\rangle - \frac{1}{2}|\xi_2\rangle, \quad |w\rangle \approx \frac{1}{2}|\xi_1\rangle + \frac{1}{2}|\xi_2\rangle$$

- From the formula for the “approximate” eigenvalues $\lambda^{\pm}(\epsilon)$, we can compute the running time:

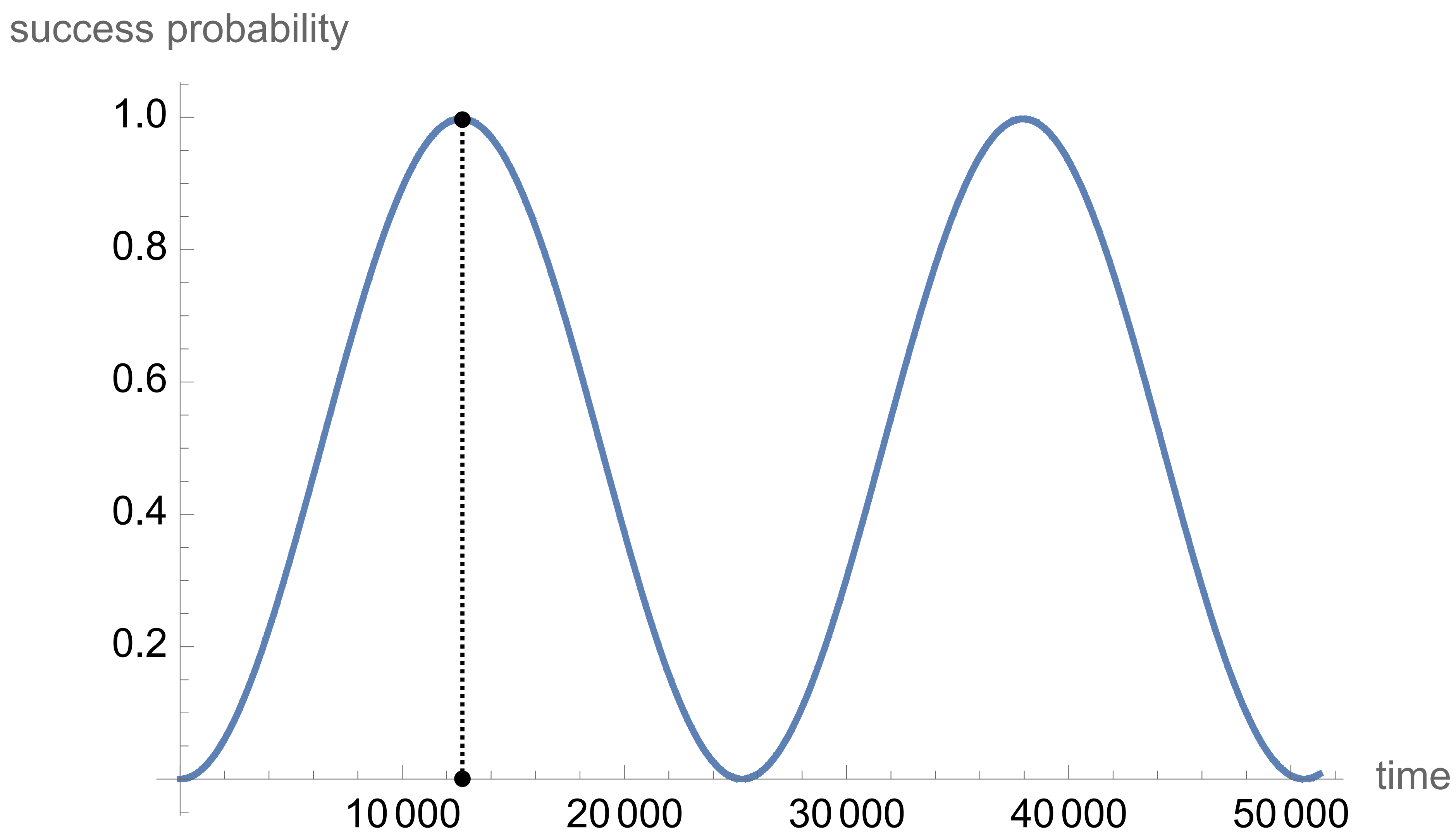
$$t = \frac{\pi}{2\sqrt{k!}\,\epsilon^k} = \frac{\pi n^{k/2}}{2\sqrt{k!}} \approx \frac{\pi\sqrt{N}}{2}$$

$N = \binom{n}{k}$

$\epsilon = 1/\sqrt{n}$

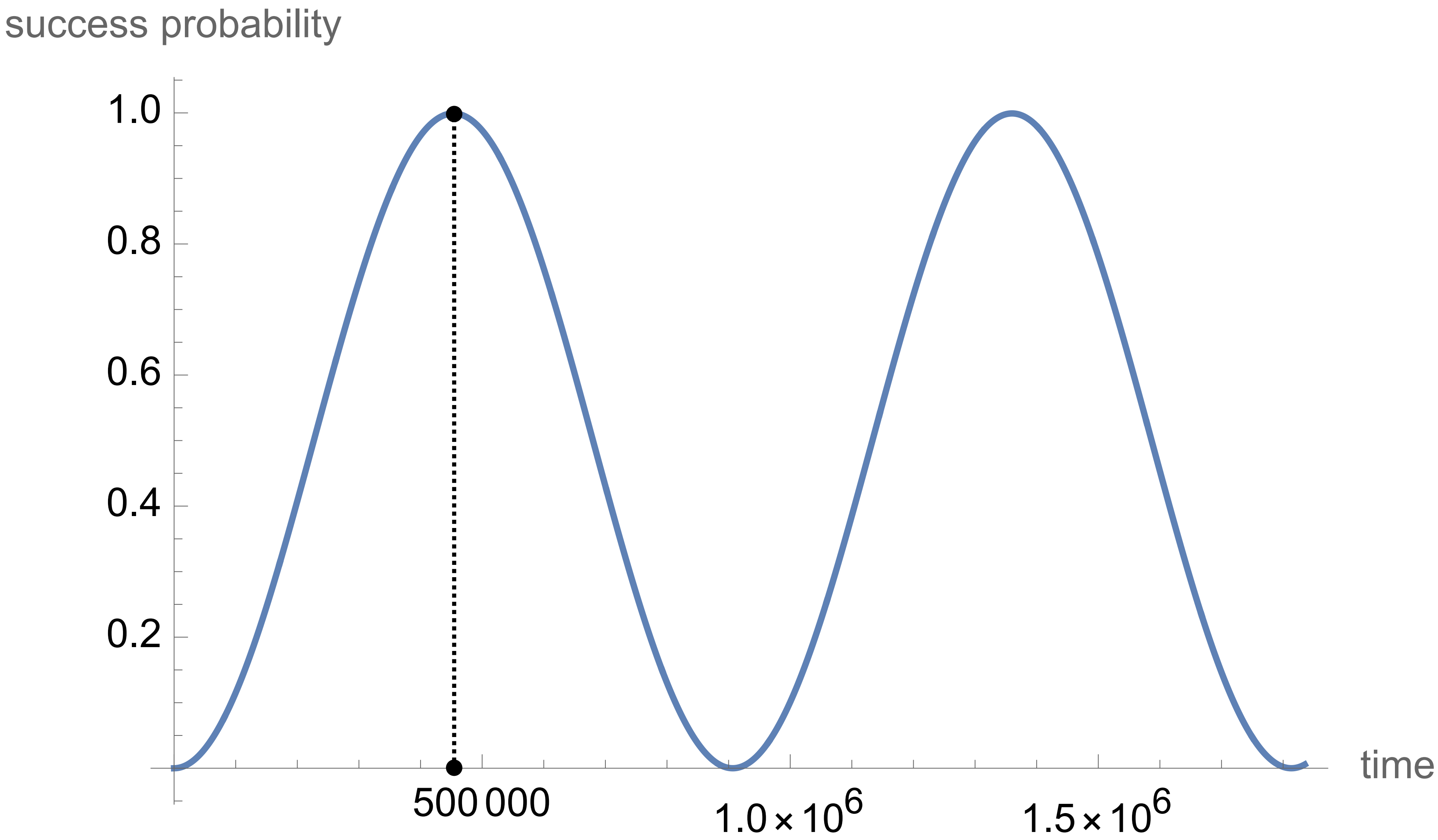
**Example** ($k = 4, n = 200$)**.**

- $t = \dfrac{\pi n^{k/2}}{2\sqrt{k!}} \approx 12{,}825.5$



## **Example** ($k = 5, n = 400$).

- $t = \dfrac{\pi n^{k/2}}{2\sqrt{k!}} \approx 458{,}859$



$$\boxed{H = -\gamma A - |w\rangle\langle w|}$$

- $\gamma = \dfrac{\epsilon^2}{\eta} = \epsilon^2 \displaystyle\sum_{\ell=1}^{k} \frac{p_\ell(\epsilon)^2}{r_0(\epsilon) - r_\ell(\epsilon)}$

$$\frac{\epsilon^2}{3} + \frac{7\epsilon^4}{6} + O(\epsilon^6)$$

$$k = 3: \quad \frac{\epsilon^2(1-3\epsilon^2)(2+\epsilon^2+16\epsilon^4-52\epsilon^6+24\epsilon^8)}{6(1-\epsilon^2)^2(1-2\epsilon^2)^2}$$

$$k = 4: \quad \frac{\epsilon^2(1-4\epsilon^2)(3-11\epsilon^2+33\epsilon^4+47\epsilon^6-660\epsilon^8+1116\epsilon^{10}-432\epsilon^{12})}{12(1-\epsilon^2)^2(1-2\epsilon^2)^2(1-3\epsilon^2)^2}$$

$$k = 5: \quad \frac{\epsilon^2(1-5\epsilon^2)(12-117\epsilon^2+532\epsilon^4-1107\epsilon^6+2508\epsilon^8-22588\epsilon^{10}+80448\epsilon^{12}-99648\epsilon^{14}+34560\epsilon^{16})}{60(1-\epsilon^2)^2(1-2\epsilon^2)^2(1-3\epsilon^2)^2(1-4\epsilon^2)^2}$$

- In the notation of Bannai–Ito, describing $\gamma$ above amounts to describing

$$\sum_{\ell=1}^{d} \frac{m_\ell}{\theta_0 - \theta_\ell}.$$

$k$

multiplicity

# Future work

- other families of DRGs, or more general graphs
- discrete-time case (see Xue–Ruan–Liu (2019) for $J(n,3)$)
- More than one marked vertex