Spatial search on Johnson graphs by discrete-time quantum walk

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Workshop on Algebraic Combinatorics

General set-up

vertex set

edge set

- $\Gamma = (\mathscr{V}, \mathscr{E})$: a finite simple graph, where $|\mathscr{V}| = N$
- $\mathscr{A} = \{(v, v') : v, v' \in \mathscr{V}, v \sim v'\}$ ← arc set

head

tail

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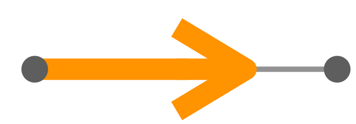

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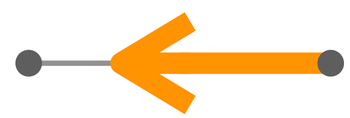

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- $\mathscr{H}_{\mathscr{A}} = \mathrm{span}\{\, |a\rangle : a \in \mathscr{A} \,\}$, where $\langle a | b \rangle = \delta_{a,b}$
- S : the **shift operator** on $\mathscr{H}_{\mathscr{A}}$

  opposite arc of a

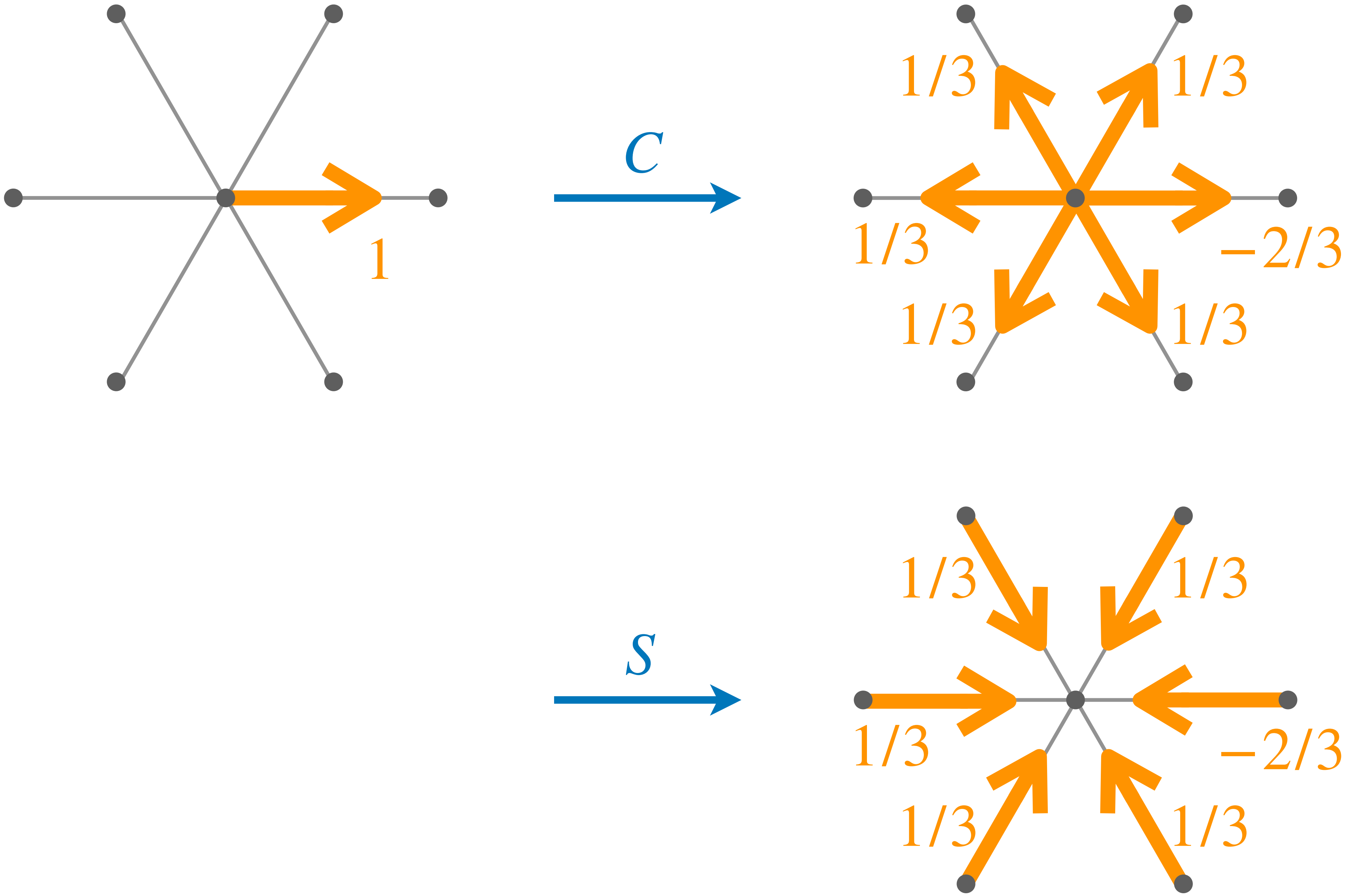
$$S|a\rangle = |\overline{a}\rangle$$

- C : the **Grover coin operator** on $\mathscr{H}_{\mathscr{A}}$

$$C|a\rangle = \frac{2}{\deg(\mathrm{tail}(a))} \sum_{\mathrm{tail}(b)=\mathrm{tail}(a)} |b\rangle - |a\rangle$$

General set-up

- $U = SC$: the **evolution operator** on $\mathscr{H}_{\mathscr{A}}$



General set-up

$U = SC$

unit vector

- $|\psi(0)\rangle \in \mathscr{H}_{\mathscr{A}}$: the initial state
- $|\psi(t)\rangle = U^t|\psi(0)\rangle$: the state at time $t \in \mathbb{N}$
- $\sum_{\mathrm{tail}(a)=v} |\langle a|\psi(t)\rangle|^2$: the probability of finding v at time t

Problem.

- $w \in \mathscr{V}$: the **marked** vertex
- Find w in time of order $\sqrt{N}$!!

General set-up

$U = SC$

Problem.

- $w \in \mathscr{V}$: the **marked** vertex
- Find w in time of order $\sqrt{N}$!!

- R : the **oracle** for w

$$R = I - 2|\tilde{w}\rangle\langle\tilde{w}|$$

where

$$|\tilde{w}\rangle = \frac{1}{\sqrt{\deg(w)}} \sum_{\mathrm{tail}(a)=w} |a\rangle$$

- $U' = UR$: the **modified evolution operator** on $\mathscr{H}_{\mathscr{A}}$

Some previous work

- complete graphs (Grover, 1996)
- hypercubes (Shenvi–Kempe–Whaley, 2003)
- finite two-dimensional lattices (Tulsi, 2008)
- Johnson graphs with diameter 3 (Xue–Ruan–Liu, 2019)

Today's topic

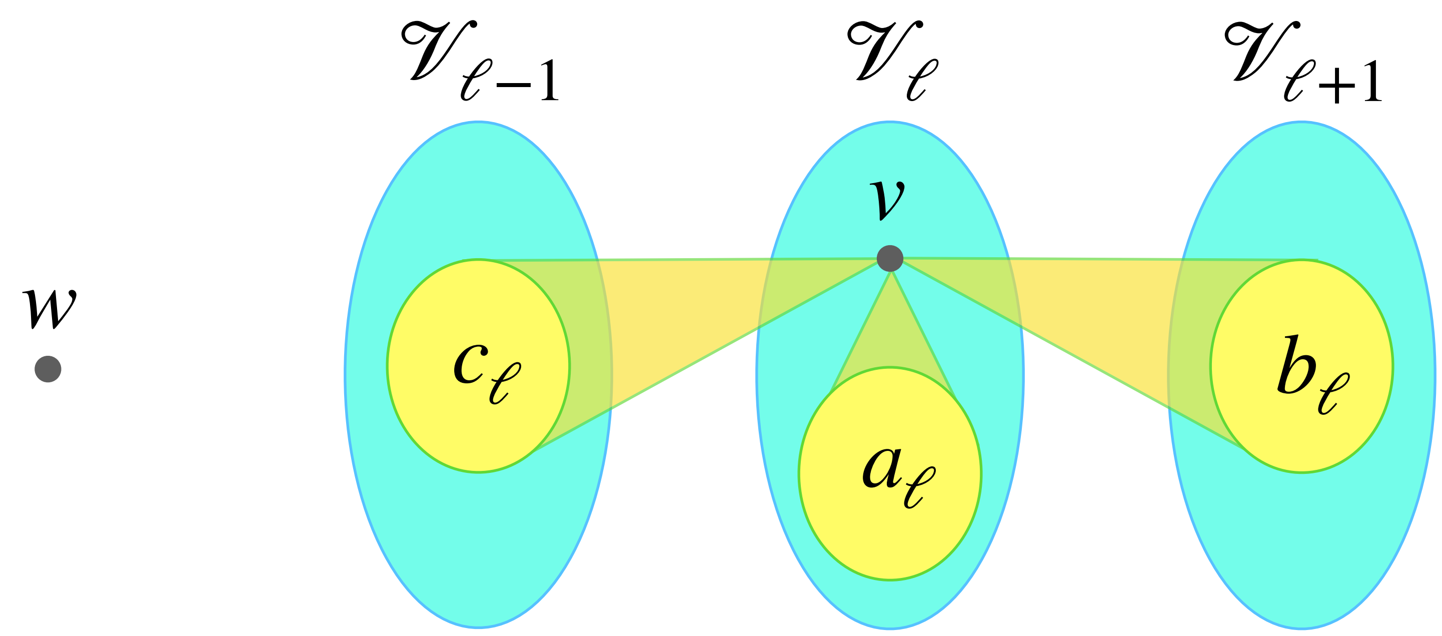
- Johnson graphs with any fixed diameter

arXiv:2112.03744

Johnson graphs

- $n, k \in \mathbb{N} \ \ (1 \leqslant k \leqslant n/2)$
- $\mathscr{V}$: the set of k-subsets of $\{1, 2, \ldots, n\} \implies N = \binom{n}{k}$
- $v \sim v' \overset{\text{def}}{\iff} |v \cap v'| = k - 1 \quad (v, v' \in \mathscr{V})$
- $\Gamma = J(n, k)$

- $\mathscr{V}_\ell = \{v \in \mathscr{V} : |v \cap w| = k - \ell\} \ \ (0 \leqslant \ell \leqslant k)$



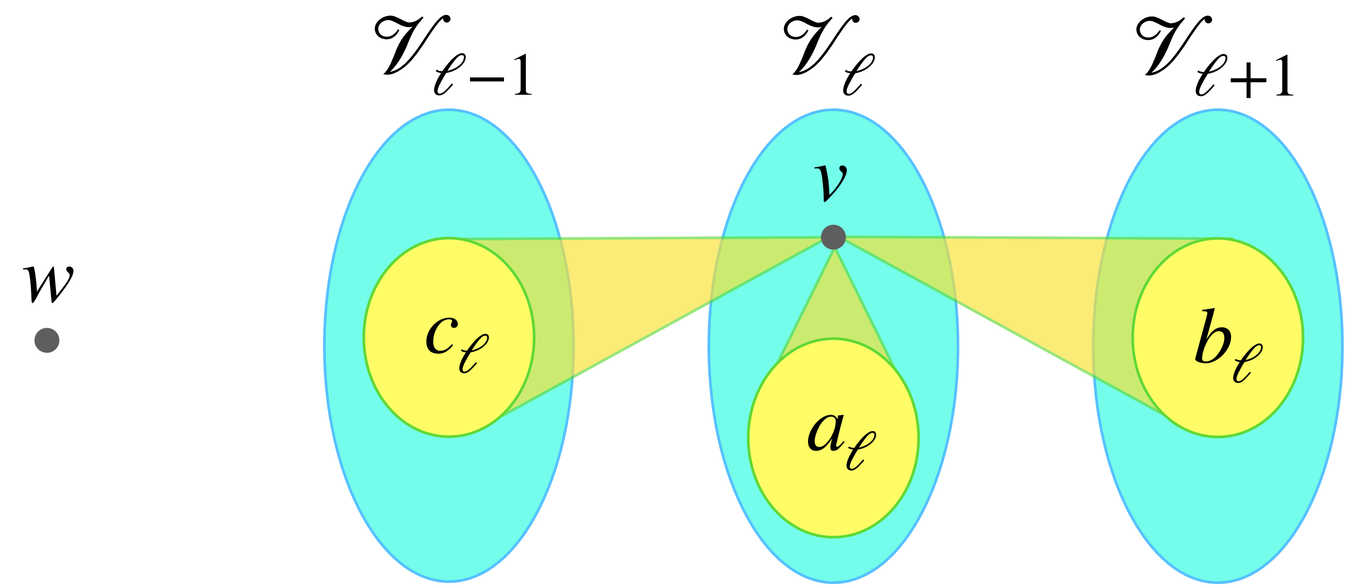
$a_\ell = \ell(n - 2\ell)$

$b_\ell = (k - \ell)(n - k - \ell)$

$c_\ell = \ell^2$

Johnson graphs

- $U' = UR$
- $R = I - 2|\tilde{w}\rangle\langle\tilde{w}|$



Note: $|\tilde{w}\rangle = \frac{1}{\sqrt{k(n-k)}}|b_0\rangle$

- $$|a_\ell\rangle = \sum_{\substack{\text{tail}(a) \in \mathscr{V}_\ell \\ \text{head}(a) \in \mathscr{V}_\ell}} |a\rangle, \quad |b_\ell\rangle = \sum_{\substack{\text{tail}(a) \in \mathscr{V}_\ell \\ \text{head}(a) \in \mathscr{V}_{\ell+1}}} |a\rangle, \quad |c_\ell\rangle = \sum_{\substack{\text{tail}(a) \in \mathscr{V}_\ell \\ \text{head}(a) \in \mathscr{V}_{\ell-1}}} |a\rangle$$

- $$\mathscr{H}_{\mathscr{A}}^{\text{inv}} = \text{span}\{|a_\ell\rangle + |b_\ell\rangle + |c_\ell\rangle, |b_\ell\rangle - |c_{\ell+1}\rangle\}_{\ell=0}^{k}$$
$$= \text{span}\{|a_\ell\rangle + |b_{\ell-1}\rangle + |c_{\ell+1}\rangle, |b_\ell\rangle - |c_{\ell+1}\rangle\}_{\ell=0}^{k}$$

invariant under S, C, U, R, U'

$\dim = 2k + 1$

The spectrum of $U = SC$

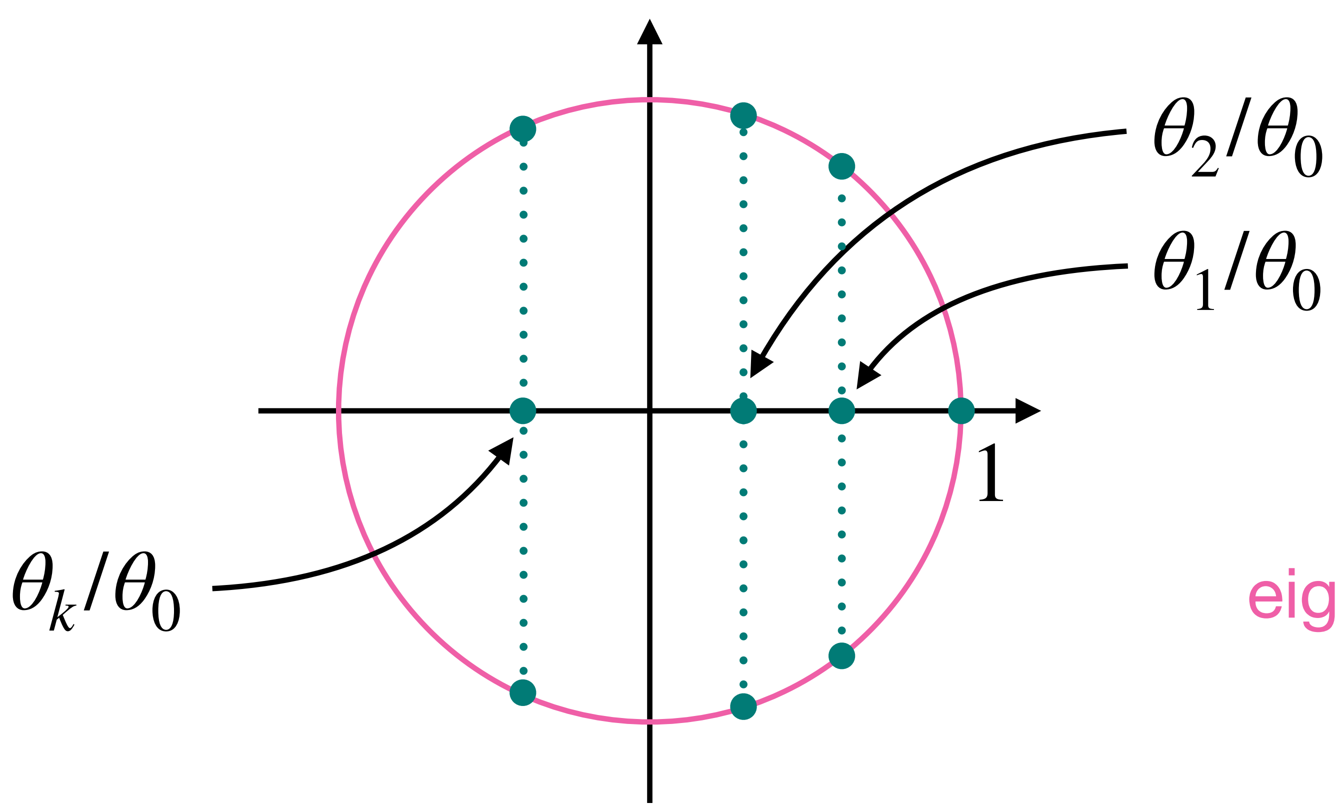
- $U' = UR$
- $R = I - 2|\tilde{w}\rangle\langle\tilde{w}|$

- $J(n,k)$ has $k+1$ distinct eigenvalues

$$\theta_\ell = (k-\ell)(n-k-\ell) - \ell \quad (0 \leqslant \ell \leqslant k).$$

- The eigenvalues of U on $\mathscr{H}_{\mathscr{A}}^{\text{inv}}$ are $1, \mathrm{e}^{\pm i\omega_1}, \ldots, \mathrm{e}^{\pm i\omega_k}$, where

$$\omega_\ell = \arccos\left(\frac{\theta_\ell}{\theta_0}\right) \quad (1 \leqslant \ell \leqslant k).$$



eigenbasis: $|\omega_0\rangle, |\omega_1^\pm\rangle, \ldots, |\omega_k^\pm\rangle$

The strategy

- $U' = UR$
- $R = I - 2|\tilde{w}\rangle\langle\tilde{w}|$

Fix k and let $n \to \infty$!!

w.r.t. $|\omega_0\rangle, |\omega_1^+\rangle, |\omega_1^-\rangle, \ldots, |\omega_k^-\rangle$

- $|\psi(0)\rangle := |\omega_0\rangle = (1,0,\ldots,0)^\mathsf{T}$
- Set $\epsilon = \dfrac{1}{\sqrt{n}}$.
- $|\tilde{w}\rangle = \dfrac{1}{\sqrt{k(n-k)}}|b_0\rangle = \dfrac{1}{\sqrt{2}}(0,\ldots,0,1,1)^\mathsf{T} + O(\epsilon)$

The strategy

- $|\psi(0)\rangle = (1,0,\ldots,0)^{\mathsf{T}}$
- $|\tilde{w}\rangle \approx \frac{1}{\sqrt{2}}(0,\ldots,0,1,1)^{\mathsf{T}}$

- $U' = UR$ has two "approximate" eigenvalues and eigenvectors:

$$U'\mathbf{x}^{\sharp} = \lambda^{\sharp}\mathbf{x}^{\sharp} + O(\epsilon^{k+1}), \quad U'\mathbf{x}^{\flat} = \lambda^{\flat}\mathbf{x}^{\flat} + O(\epsilon^{k+1}),$$

where

$$\lambda^{\sharp} = 1 - \mathrm{i}\sqrt{2k!}\,\epsilon^{k} + O(\epsilon^{k+1}),$$
$$\lambda^{\flat} = 1 + \mathrm{i}\sqrt{2k!}\,\epsilon^{k} + O(\epsilon^{k+1}),$$

$$\mathbf{x}^{\sharp} = (1-\mathrm{i},0,\ldots,0,1,\mathrm{i})^{\mathsf{T}} + O(\epsilon),$$

$$\mathbf{x}^{\flat} = (\mathrm{i}-1,0,\ldots,0,1,\mathrm{i})^{\mathsf{T}} + O(\epsilon).$$

The strategy

- $\lambda^\sharp = 1 - \mathrm{i}\sqrt{2k!}\epsilon^k + O(\epsilon^{k+1})$
- $\lambda^\flat = 1 + \mathrm{i}\sqrt{2k!}\epsilon^k + O(\epsilon^{k+1})$

$$|\psi(0)\rangle = (1,0,\ldots,0)^\mathsf{T}$$
$$|\tilde{w}\rangle \approx \frac{1}{\sqrt{2}}(0,\ldots,0,1,1)$$

$$\mathbf{x}^\sharp \approx (1-\mathrm{i},0,\ldots,0,1,\mathrm{i})^\mathsf{T}$$
$$\mathbf{x}^\flat \approx (\mathrm{i}-1,0,\ldots,0,1,\mathrm{i})^\mathsf{T}$$

- $|\psi(0)\rangle \approx \frac{1+\mathrm{i}}{4}(\mathbf{x}^\sharp - \mathbf{x}^\flat) \in \mathrm{span}\{\mathbf{x}^\sharp, \mathbf{x}^\flat\}$
- $|\psi(t)\rangle \approx \frac{1+\mathrm{i}}{4}\left((\lambda^\sharp)^t\mathbf{x}^\sharp - (\lambda^\flat)^t\mathbf{x}^\flat\right) \in \mathrm{span}\{\mathbf{x}^\sharp, \mathbf{x}^\flat\}$
- $|\tilde{w}\rangle|_{\mathrm{span}\{\mathbf{x}^\sharp,\mathbf{x}^\flat\}} = \frac{1-\mathrm{i}}{4\sqrt{2}}(\mathbf{x}^\sharp + \mathbf{x}^\flat)$
- $t_{\mathrm{run}} := \left\lfloor \frac{\pi}{2\sqrt{2k!}\,\epsilon^k} \right\rfloor \approx \frac{\pi\sqrt{N}}{2\sqrt{2}} \implies \left(\frac{\lambda^\flat}{\lambda^\sharp}\right)^{t_{\mathrm{run}}} = -1 + O(\epsilon)$

The strategy

- $t_{\text{run}} \approx \dfrac{\pi\sqrt{N}}{2\sqrt{2}}$
- $\left(\dfrac{\lambda^\flat}{\lambda^\sharp}\right)^{t_{\text{run}}} \approx -1$

- $|\psi(0)\rangle \approx \frac{1+\mathrm{i}}{4}(\mathbf{x}^\sharp - \mathbf{x}^\flat)$
- $$\begin{aligned} |\psi(t_{\text{run}})\rangle &\approx \frac{1+\mathrm{i}}{4}\left((\lambda^\sharp)^{t_{\text{run}}}\mathbf{x}^\sharp - (\lambda^\flat)^{t_{\text{run}}}\mathbf{x}^\flat\right) \\ &= \frac{1+\mathrm{i}}{4}(\lambda^\sharp)^{t_{\text{run}}}\left(\mathbf{x}^\sharp - \left(\frac{\lambda^\flat}{\lambda^\sharp}\right)^{t_{\text{run}}}\mathbf{x}^\flat\right) \\ &\approx \frac{1+\mathrm{i}}{4}(\lambda^\sharp)^{t_{\text{run}}}(\mathbf{x}^\sharp + \mathbf{x}^\flat) \\ &\approx \frac{1+\mathrm{i}}{2}(\lambda^\sharp)^{t_{\text{run}}}(0,\ldots,0,1,\mathrm{i}) \end{aligned}$$
- The success probability is

$$\sum_{\text{tail}(a)=w} |\langle a|\psi(t_{\text{run}})\rangle|^2 = |\langle \tilde{w}|\psi(t_{\text{run}})\rangle|^2 = \frac{1}{2} + O(\epsilon).$$

The strategy

Theorem (Implicit Function Theorem)**.**

- $\varphi_i(\boldsymbol{x}, \boldsymbol{y})$ $(1 \leqslant i \leqslant q)$: analytic functions $(\boldsymbol{x} \in \mathbb{C}^p, \boldsymbol{y} \in \mathbb{C}^q)$
- $\varphi_i(\boldsymbol{x}^0, \boldsymbol{y}^0) = 0$ $(1 \leqslant i \leqslant q)$
- $\det\left(\dfrac{\partial \varphi_j}{\partial y_i}\right)_{i,j=1}^{q}\Bigg|_{(\boldsymbol{x}^0, \boldsymbol{y}^0)} \neq 0$

$\Longrightarrow$ ① $\exists U$: a neighborhood of $(\boldsymbol{x}^0, \boldsymbol{y}^0)$

② $g_1(\boldsymbol{x}), \ldots, g_q(\boldsymbol{x})$: analytic functions

s.t., on U,

$$\varphi_i(\boldsymbol{x}, \boldsymbol{y}) = 0 \ (1 \leqslant i \leqslant q) \iff \boldsymbol{y} = (g_1(\boldsymbol{x}), \ldots, g_q(\boldsymbol{x}))$$

The continuous-time case

adjacency matrix

- $\mathscr{H}_{\mathscr{V}} = \mathrm{span}\{\,|v\rangle : v \in \mathscr{V}\,\}$, where $\langle v\,|\,v'\rangle = \delta_{v,v'}$
- $H = -\gamma A - |w\rangle\langle w|$: a **Hamiltonian** where $\gamma \in \mathbb{R}$
- $|\psi(0)\rangle \in \mathscr{H}_{\mathscr{V}}$: the initial state
- $|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht}|\psi(0)\rangle$: the state at time $t \in \mathbb{R}$
- $|\langle w\,|\,\psi(t)\rangle|^2$: the probability of finding w at time t

Remark.

- $H, \mathrm{e}^{-\mathrm{i}Ht} \in$ the Terwilliger algebra w.r.t. w
- $J(n,3)$ (Wong, 2016)
- $J(n,k)$ with any fixed k (T.–Sabri–Portugal, 2021)

Future work

- other families of DRGs, or more general graphs
- more than one marked vertex
- the **element distinctness problem** (Ambainis, 2007)