On Some Relationships Among the Association Schemes of Finite Orthogonal Groups Acting on Hyperplanes

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Introduction

An *association scheme* $\mathfrak{X} = (X, \{R_i\}_{0\le i\le d})$ *of class* d is a pair of a finite set X and a set of nontrivial relations $\{R_i\}_{0\le i\le d}$ on X satisfying the following four conditions:

(i) $R_0 = \{(x,x) \in X \times X \mid x \in X\}$,

(ii) $R_0 \cup R_1 \cup \cdots \cup R_d = X \times X$ and $R_i \cap R_j = \emptyset$ if $i \neq j$,

(iii) for each $i \in \{0,1,\ldots,d\}$, there exists some $i' \in \{0,1,\ldots,d\}$ such that ${}^tR_i = R_{i'}$ holds, where ${}^tR_i := \{(y,x) \in X \times X \mid (x,y) \in R_i\}$,

(iv) for each (orderd) triple $i,j,k \in \{0,1,\ldots,d\}$, the cardinality of the set $\{z \in X \mid (x,z) \in R_i,\ (z,y) \in R_j\}$, which is denoted by p_{ij}^k, does not depend on the choice of $x,y \in X$ under the condition $(x,y) \in R_k$.

The numbers p_{ij}^k in condition (iv) are called the *intersection numbers* of $\mathfrak{X} = (X, \{R_i\}_{0\le i\le d})$, and in particular we call the numbers $k_i := p_{ii'}^0 = \big|\{z \in X \mid (x,z) \in R_i\}\big|$ $(0 \le i \le d)$ the *valencies* of $\mathfrak{X} = (X, \{R_i\}_{0\le i\le d})$.

Let A_i be the adjacency matrix with respect to the relation R_i, that is,

$$(A_i)_{x,y} := \begin{cases} 1 & \text{if } (x,y) \in R_i \\ 0 & \text{if } (x,y) \notin R_i, \end{cases}$$

then, since $A_iA_j = \sum_{k=0}^d p_{ij}^k A_k$ by condition (iii), $A_0, A_1, \ldots, A_d$ generates an algebra $\mathfrak{A}$ over the complex field $\mathbb{C}$ of dimension $d+1$. We call this algebra the *Bose-Mesner algebra* of the association scheme. We say that an association scheme $\mathfrak{X} = (X, \{R_i\}_{0\le i\le d})$ is *commutative*, if the Bose-Mesner algebra is commutative, or equivalently, if $p_{ij}^k = p_{ji}^k$ holds for all $i,j,k \in \{0,1,\ldots,d\}$. A *symmetric* association scheme is an association scheme $\mathfrak{X} = (X, \{R_i\}_{0\le i\le d})$ which satisfies ${}^tR_i = R_i$ for all $i \in \{0,1,\ldots,d\}$. Notice that a symmetric association scheme is a commutative association scheme.

Let $\mathfrak{X} = (X, \{R_i\}_{0\le i\le d})$ be a commutative association scheme, then the Bose-Mesner algebra has a unique set of primitive idempotents $E_0 = \frac{1}{|X|}J, E_1, \ldots, E_d$, where J is the matrix whose entries are all 1 (cf. Bannai-Ito [4, §2.3.]). Let

$$A_i = \sum_{j=0}^d p_i(j)E_j.$$

Then the $(d+1)$ by $(d+1)$ matrix P whose (j,i)-entry is $p_i(j)$, is called the *character table* or the *first eigenmatrix* of the association scheme. The character table P of an association scheme $\mathfrak{X} = (X, \{R_i\}_{0\le i\le d})$ satisfies the *orthogonality relations* (cf. Bannai-Ito [4, p.62, Theorem 3.5.]):

(i) (The First Orthogonality Relation)

$$\sum_{\alpha=0}^d \frac{1}{k_\alpha} p_\alpha(i)\overline{p_\alpha(j)} = \frac{|X|}{m_i}\delta_{ij},$$

(ii) (The Second Orthogonality Relation)

$$\sum_{\alpha=0}^d m_\alpha p_i(\alpha)\overline{p_j(\alpha)} = |X|k_i\delta_{ij},$$

where $m_i := \operatorname{rank} E_i = \operatorname{tr} E_i$ $(0 \le i \le d)$, and δ_{ij} is the Kronecker delta. The numbers m_i are called the *multiplicities* of $\mathfrak{X} = (X, \{R_i\}_{0\le i\le d})$. In particular we shall use the following equality:

$$\sum_{\alpha=0}^d p_\alpha(i) = 0, \quad \text{if } 1 \le i \le d. \tag{1}$$

A *subassociation scheme* (or simply *subscheme*) of an association scheme $\mathfrak{X} = (X, \{R_i\}_{0\leq i\leq d})$ is an association scheme $\mathfrak{X}' = (X, \{S_j\}_{0\leq j\leq d'})$ where each relation S_j, $j \in \{0, 1, \ldots, d'\}$ is a union of some R_i's. It is an interesting problem to find all the subschemes of an association scheme. Bannai [1, Lemma 1.] showed that any subscheme of a given commutative association scheme is obtained by partitioning its character table into appropriate blocks.

It is natural to regard association schemes as a combinatorial interpretation of finite transitive permutation groups. Let G be a finite group acting transitively on a finite set X. Then G acts naturally on $X \times X$ in such a way that

$$(x, y)^g := (x^g, x^g) \text{ for } (x, y) \in X \times X,\ g \in G,$$

and we can easily verify that the orbits of G acting on $X \times X$ (which are called the *orbitals*) satisfy the above four conditions (cf. Bannai-Ito [4, p.53, Example 2.1.]), that is, the action of G on $X \times X$ defines an association scheme. We denote this association scheme by $\mathfrak{X}(G, X)$. It is well known that $\mathfrak{X}(G, X)$ is commutative if and only if the permutation character 1_H^G is *multiplicity-free* where H is the stabilizer of an element of X, namely each irreducible character of G occurs in the decomposition with multiplicity at most 1 (cf. Bannai-Ito [4, p.49, Theorem 1.4.]). If $\mathfrak{X}(G, X)$ is commutative, then determining the character table of $\mathfrak{X}(G, X)$ is equivalent to determining all the zonal spherical functions of G on X (cf. Bannai-Ito [4, §2.11.]).

In this paper, we study the association schemes defined by the action of the orthogonal groups $GO_{2m+1}(q)$ over the finite fields of characteristic 2, on the set $\Omega = \Omega_{2m+1}(q)$ of positive-type hyperplanes and on the set $\Theta = \Theta_{2m+1}(q)$ of negative-type hyperplanes. These association schemes are isomorphic to the association schemes defined by the action of $GO_{2m+1}(q)$ on the set of cosets by $GO^+_{2m}(q)$ and on the set of cosets by $GO^-_{2m}(q)$, respectively.

This paper is organized as follows:

Introduction.

1. Preliminary;
   1.1. Quadratic forms and orthogonal groups.
   1.2. Description of the relations.
2. Computation of parameters;
   2.1. The parameters of $\mathfrak{X}(GO_{2m+1}(q), \Omega_{2m+1}(q))$.
   2.2. The parameters of $\mathfrak{X}(GO_{2m+1}(q), \Theta_{2m+1}(q))$.
3. Character tables;
   3.1. The character tables of $\mathfrak{X}(GO_{2m+1}(q), \Omega_{2m+1}(q))$.
   3.2. The character tables of $\mathfrak{X}(GO_{2m+1}(q), \Theta_{2m+1}(q))$.
4. Subschemes;
   4.1. Subschemes of $\mathfrak{X}(GO_{2m+1}(q), \Omega_{2m+1}(q))$.
   4.2. Subschemes of $\mathfrak{X}(GO_{2m+1}(q), \Theta_{2m+1}(q))$.
5. Remarks.

In section 3, we calculate the character tables of these association schemes. In fact, we will show that the character tables of $\mathfrak{X}(GO_{2m+1}(q), \Omega_{2m+1}(q))$ and $\mathfrak{X}(GO_{2m+1}(q), \Theta_{2m+1}(q))$ are *controlled* by the character tables of $\mathfrak{X}(GO_3(q), \Omega_3(q))$ and $\mathfrak{X}(GO_3(q), \Theta_3(q))$, respectively, by the replacement $q \to q^{m-1}$. It is known that such phenomena occur in many cases (cf. Bannai-Hao-Song [2], Bannai-Hao-Song-Wei [3], Kwok [9], Bannai-Kawanaka-Song [5]). Our method of calculating character tables follows Bannai-Hao-Song [2, §6,7] in all essential points, where they determined the character tables of the association schemes obtained from the action of finite orthogonal groups on the sets of non-isotropic projective points. Actually, the association schemes treated in this paper correspond to the case of even q.

In section 4, we first show that $\mathfrak{X}(GO_{2m+1}(q), \Omega_{2m+1}(q))$ and $\mathfrak{X}(GO_{2m+1}(q), \Theta_{2m+1}(q))$ are subschemes of $\mathfrak{X}(GO_3(q^m), \Omega_3(q^m))$ and $\mathfrak{X}(GO_3(q^m), \Theta_3(q^m))$, respectively. Then we write down all the relations of these subschemes from those of $\mathfrak{X}(GO_3(q^m), \Omega_3(q^m))$ and $\mathfrak{X}(GO_3(q^m), \Theta_3(q^m))$. It is also shown that $\mathfrak{X}(GO_{2n+1}(q), \Omega_{2n+1}(q))$ is a subscheme of $\mathfrak{X}(GO_{2m+1}(q^{\frac{n}{m}}), \Omega_{2m+1}(q^{\frac{n}{m}}))$ whenever m divides n, and so forth.

Thus we can say that the two association schemes $\mathfrak{X}(GO_3(q), \Omega_3(q))$ and $\mathfrak{X}(GO_3(q), \Theta_3(q))$ controll the other association schemes $\mathfrak{X}(GO_{2m+1}(q), \Omega_{2m+1}(q))$ and $\mathfrak{X}(GO_{2m+1}(q), \Theta_{2m+1}(q))$ at two levels—*algebraic level and combinatorial level*.

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1 Preliminary

1.1 Quadratic Forms and Orthogonal Groups

In this subsection, we review some basic facts on quadratic forms and orthogonal groups. For more information, we are referred to Munemasa [12], **ATLAS** [8].

Let $\mathbb{V}$ be a finite dimensional vector space over the finite field $\mathbb{F}_q$ of q elements. A *symmetric bilinear form* on $\mathbb{V}$ over $\mathbb{F}_q$ is a mapping $f : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{F}_q$ which satisfies the following conditions:

$$f(u_1 + u_2, v) = f(u_1, v) + f(u_2, v),$$
$$f(\alpha u, v) = \alpha f(u, v),$$
$$f(u, v) = f(v, u)$$

for all $u, v, u_1, u_2 \in \mathbb{V}$ and all $\alpha \in \mathbb{F}_q$. We define the *orthogonal complement* $U^\perp$ of a subset U of $\mathbb{V}$ by

$$U^\perp := \{v \in \mathbb{V} \mid f(u, v) = 0 \text{ for all } u \in U\},$$

and the *radical* of f by

$$\operatorname{Rad} f := \mathbb{V}^\perp = \{v \in \mathbb{V} \mid f(u, v) = 0 \text{ for all } u \in \mathbb{V}\}.$$

The symmetric bilinear form f is said to be *non-degenerate* if $\operatorname{Rad} f = 0$. The following proposition is a basic fact about non-degenerate symmetric bilinear forms (cf. Munemasa [12, p.3, Proposition 1.1.]).

Proposition 1.1.1. *Let $f : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{F}_q$ be a symmetric bilinear form on a finite dimensional vector space $\mathbb{V}$ over $\mathbb{F}_q$, and let U be a subspace of $\mathbb{V}$. Then we have*

$$\dim U^\perp = \dim \mathbb{V} - \dim U + \dim U \cap \operatorname{Rad} f.$$

Moreover if $f|_U$ is non-degenerate then

$$\mathbb{V} = U \perp U^\perp.$$

A *quadratic form* on $\mathbb{V}$ over $\mathbb{F}_q$ is a mapping $Q : \mathbb{V} \longrightarrow \mathbb{F}_q$ which satisfies the following conditions:

$$Q(\alpha v) = \alpha^2 Q(v),$$
$$Q(u + v) = Q(u) + Q(v) + f(u, v)$$

for all $u, v \in \mathbb{V}$ and all $\alpha \in \mathbb{F}_q$, where $f : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{F}_q$ is a symmetric bilinear form on $\mathbb{V}$ over $\mathbb{F}_q$. Notice that if q is even, then the bilinear form f is an *alternating* bilinear form, that is, $f(v, v) = 0$ for all $v \in \mathbb{V}$. The quadratic form Q is said to be *non-degenerate* if $Q^{-1}(0) \cap \operatorname{Rad} f = \{0\}$. If a vector $v \in \mathbb{V}$ satisfies $Q(v) = 0$, then we call this vector *singular*, and a subspace U of $\mathbb{V}$ which consists of sigular vectors is also called *singular*. A *hyperbolic pair* is a pair of vectors $\{u, v\}$ of $\mathbb{V}$ satisfying $Q(u) = Q(v) = 0$, and $f(u, v) = 1$. For later use, we need the following proposition (cf. Munemasa [12, p.7, Proposition 1.8.]).

Proposition 1.1.2. *Let $Q : \mathbb{V} \longrightarrow \mathbb{F}_q$ be a non-degenerate quadratic form on a finite dimensional vector space $\mathbb{V}$ over $\mathbb{F}_q$ and let $u \in \mathbb{V}$ be a non-zero singular vector. Then there exists a vector $v \in \mathbb{V}$ such that $\{u, v\}$ is a hyperbolic pair.*

The *orthogonal group* $O(\mathbb{V}, Q)$ is the group which consists of all automorphisms of Q. More precisely,

$$O(\mathbb{V}, Q) := \{\tau \in GL(\mathbb{V}) \mid Q(\tau(v)) = Q(v) \text{ for all } v \in \mathbb{V}\}.$$

Throughout this paper, we always assume that q is even. Let Q be a non-degenerate quadratic form on $\mathbb{V}$. Suppose $\dim \mathbb{V} = 2m+1$ is odd, then there exists a basis $\{v_1, v_2, \ldots, v_{2m+1}\}$ of $\mathbb{V}$ such that

$$Q\Big(\sum_{i=1}^{2m+1} \xi_i v_i\Big) = \xi_1\xi_{m+1} + \xi_2\xi_{m+2} + \cdots + \xi_m\xi_{2m} + \xi_{2m+1}^2,$$

which is equivalent to saying that $\mathbb{V}$ is decomposed as

$$\mathbb{V} = \langle v_1, v_{m+1}\rangle \perp \ldots \perp \langle v_m, v_{2m}\rangle \perp \langle v_{2m+1}\rangle,$$

where $\{v_1, v_{m+1}\}, \ldots, \{v_m, v_{2m}\}$ are hyperbolic pairs and $Q(v_{2m+1}) = 1$. We write $GO_{2m+1}(q) = O(\mathbb{V}, Q)$. Suppose $\dim \mathbb{V} = 2m$ is even, then one of the following occurs:

(i) there exists a basis $\{v_1, v_2, \ldots, v_{2m}\}$ of $\mathbb{V}$ such that

$$Q\Big(\sum_{i=1}^{2m} \xi_i v_i\Big) = \xi_1\xi_{m+1} + \xi_2\xi_{m+2} + \cdots + \xi_m\xi_{2m},$$

(ii) there exists a basis $\{v_1, v_2, \ldots, v_{2m}\}$ of $\mathbb{V}$ such that

$$Q\Big(\sum_{i=1}^{2m} \xi_i v_i\Big) = \xi_1\xi_{m+1} + \xi_2\xi_{m+2} + \cdots + \xi_{m-1}\xi_{2m-1} + \xi_m^2 + \xi_m\xi_{2m} + \pi\xi_{2m}^2,$$

where $t^2 + t + \pi$ is an irreducible polynomial over $\mathbb{F}_q$. In what follows, we call the former *positive-type* and the latter *negative-type*, and we write their orthogonal groups as $GO^+_{2m}(q)$ and $GO^-_{2m}(q)$, respectively.

We end this subsection by proving the following enumerative lemma (cf. Bannai-Hao-Song [2, Lemma 1.1.]).

Lemma 1.1.3. *For $\beta \in \mathbb{F}_q$ and a polynomial $h(\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{F}_q[\xi_1, \xi_2, \ldots, \xi_n]$, denote the number of solutions of the equation $h(\xi_1, \xi_2, \ldots, \xi_n) = \beta$ in $\mathbb{F}_q^n$ by $N[\,h(\xi_1, \xi_2, \ldots, \xi_n) = \beta\,]$.*
(i) If we denote

$$\Gamma_\beta(2m+1) := N[\,\xi_1\xi_{m+1} + \xi_2\xi_{m+2} + \cdots + \xi_m\xi_{2m} + \xi_{2m+1}^2 = \beta\,],$$

then

$$\Gamma_\beta(2m+1) = q^{2m} \ \textit{ for all } \beta \in \mathbb{F}_q.$$

(ii) If we denote

$$\Gamma^+_\beta(2m) := N[\,\xi_1\xi_{m+1} + \xi_2\xi_{m+2} + \cdots + \xi_m\xi_{2m} = \beta\,],$$

then

$$\Gamma^+_\beta(2m) = \begin{cases} q^m + q^{m-1}(q^m - 1) & \textit{for } \beta = 0 \\ q^{m-1}(q^m - 1) & \textit{for } \beta \in \mathbb{F}_q^*. \end{cases}$$

(iii) If we denote

$$\Gamma^-_\beta(2m) := N[\,\xi_1\xi_{m+1} + \xi_2\xi_{m+2} + \cdots + \xi_{m-1}\xi_{2m-1} + \xi_m^2 + \xi_m\xi_{2m} + \pi\xi_{2m}^2 = \beta\,],$$

then

$$\Gamma^-_\beta(2m) = \begin{cases} q^{m-1} + q^m(q^{m-1} - 1) & \textit{for } \beta = 0 \\ q^{m-1}(q^m + 1) & \textit{for } \beta \in \mathbb{F}_q^*, \end{cases}$$

where $t^2 + t + \pi$ is an irreducible polynomial over $\mathbb{F}_q$.

Proof. (i) Since $\mathbb{F}_q$ is assumed to be characteristic 2, any element in $\mathbb{F}_q$ is a square. Thus we can choose $\xi_1, \xi_2, \dots, \xi_{2m}$ arbitrarily. (ii) Suppose $(\xi_{m+1}, \xi_{m+2}, \dots, \xi_{2m}) = (0, 0, \dots, 0)$, then if $\beta = 0$ we have q^m choices for $\xi_1, \xi_2, \dots, \xi_m$. Next suppose $(\xi_{m+1}, \xi_{m+2}, \dots, \xi_{2m}) \neq (0, 0, \dots, 0)$, say $\xi_{2m} \neq 0$. Then ξ_m is uniquely determined depending on $\xi_1, \xi_2, \dots, \xi_{m-1}$, hence we have q^{m-1} choices for $\xi_1, \xi_2, \dots, \xi_m$. (iii) First we consider the case $m = 1$. If $\xi_2 = 0$, then clearly ξ_1 is uniquely determined. If $\xi_2 \neq 0$, then the number of solutions of the equation

$$\xi_1^2 + \xi_1\xi_2 + \pi\xi_2^2 = \beta, \quad \xi_2 \neq 0$$

is equal to the number of solutions of the equation

$$\xi^2(\eta^2 + \eta + \pi) = \beta, \quad \xi \neq 0,$$

by putting $\xi := \xi_2$ and $\eta := \frac{\xi_1}{\xi_2}$. Since $t^2 + t + \pi$ is irreducible over $\mathbb{F}_q$, if $\beta = 0$ then there is no solution, and if $\beta \neq 0$ then there are exactly q solutions. Thus, we have

$$\Gamma_\beta^-(2) = \begin{cases} 1 & \text{for } \beta = 0 \\ q+1 & \text{for } \beta \in \mathbb{F}_q^*. \end{cases}$$

Consequently, from (ii) we have

$$\begin{aligned} \Gamma_0^-(2m) &= \{q^{m-1} + q^{m-2}(q^{m-1} - 1)\} + (q-1)(q+1)q^{m-2}(q^{m-1} - 1) \\ &= q^{m-1} + q^m(q^{m-1} - 1), \end{aligned}$$

and for $\beta \neq 0$ we have

$$\begin{aligned} \Gamma_\beta^-(2m) &= q^{m-2}(q^{m-1} - 1) + (q+1)\{q^{m-1} + q^{m-2}(q^{m-1} - 1)\} + (q-2)(q+1)q^{m-2}(q^{m-1} - 1) \\ &= (q+1)q^{m-1} + q^m(q^{m-1} - 1) \\ &= q^{m-1}(q^m + 1), \end{aligned}$$

which completes the proof of Lemma 1.1.3. □

1.2 Description of the Relations

Let $\mathbb{V}$ be a $(2m+1)$-dimensional vector space over a finite field $\mathbb{F}_q$ of characteristic 2, and let $Q : \mathbb{V} \longrightarrow \mathbb{F}_q$ be a non-degenerate quadratic form on $\mathbb{V}$ over $\mathbb{F}_q$ with associated alternating form $f : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{F}_q$. In this case $\mathrm{Rad}\, f$ is a 1-dimensional subspace of $\mathbb{V}$, and there exists a vector $r \in \mathbb{V}$ such that $Q(r) = 1$ and

$$\mathrm{Rad}\, f = \langle r \rangle.$$

Let $U \subset \mathbb{V}$ be a subspace of $\mathbb{V}$. If the restriction of Q to U is non-degenerate (resp. degenerate), then we call this subspace *non-degenerate* (resp. *degenerate*). Moreover suppose that $\dim U$ is even, then if the restriction of Q to U is positive-type (resp. negative-type), then we call this subspace *positive-type* (resp. *negative-type*).

Denote the set of positive-type hyperplanes of $\mathbb{V}$ and the set of negative-type hypeplanes of $\mathbb{V}$ by $\Omega = \Omega_{2m+1}(q)$ and $\Theta = \Theta_{2m+1}(q)$, respectively. The orthogonal group $GO_{2m+1}(q)$ acts transitively on Ω and Θ, and the stabilizer of an element of Ω (resp. Θ) in $GO_{2m+1}(q)$ is isomorphic to $GO_{2m}^+(q)$ (resp. $GO_{2m}^-(q)$). Note that to see the transitivity we do not need the *Witt's extension theorem* (cf. Munemasa [12]), since for any $U, U' \in \Omega$ (resp. Θ), any *isometry* $\tau : U \longrightarrow U'$ (that is, τ is an injective linear map which has the property that $Q(\tau(u)) = Q(u)$ for all $u \in U$) is extended to an automorphism $\tilde{\tau} : \mathbb{V} \longrightarrow \mathbb{V}$ by $\tilde{\tau}(r) := r$.

The numbers of positive-type and negative-type hyperplanes are given as follows (cf. **ATLAS** [8, p.xii]):

$$\begin{aligned} |\Omega| &= \frac{|GO_{2m+1}(q)|}{|GO_{2m}^+(q)|} \\ &= \frac{q^{m^2}(q^{2m} - 1)(q^{2m-2} - 1)\dots(q^2 - 1)}{2q^{m(m-1)}(q^m - 1)(q^{2m-2} - 1)\dots(q^2 - 1)} \\ &= \frac{q^m(q^m + 1)}{2}, \end{aligned} \tag{2}$$

and

$$\begin{aligned}
|\Theta| &= \frac{|GO_{2m+1}(q)|}{|GO^-_{2m}(q)|} \\
&= \frac{q^{m^2}(q^{2m}-1)(q^{2m-2}-1)\dots(q^2-1)}{2q^{m(m-1)}(q^m+1)(q^{2m-2}-1)\dots(q^2-1)} \\
&= \frac{q^m(q^m-1)}{2}. \qquad (3)
\end{aligned}$$

Now, we describe the relations of the association scheme $\mathfrak{X}(GO_{2m+1}(q),\Omega)$, defined by the action of $GO_{2m+1}(q)$ on the set Ω. Let U, V be two distinct elements in Ω. Note that $U \cap V$ is a $(2m-1)$-dimensional subspace in $\mathbb{V}$.

(i) Suppose $U \cap V$ is a degenerate subspace in $\mathbb{V}$. Then there exists a singular vector w in $U \cap V$ such that

$$U \cap V = \langle w\rangle^{\perp} \cap U = \langle w\rangle^{\perp} \cap V,$$

since $0 \subsetneq \mathrm{Rad}\, f|_{U\cap V} \subset (U\cap V)^{\perp} \cap U$ and $\dim\{(U\cap V)^{\perp}\cap U\} = \dim U - \dim(U\cap V) = 1$ by Proposition 1.1.1. Let u be a vector in U such that $\{u, w\}$ is a hyperbolic pair (Proposition 1.1.2). Then since U and V are both positive-type, there exists a positive-type hyperplane W of $U \cap V$ and a vector $v \in V$ such that $\{v, w\}$ is a hyperbolic pair and

$$U = \langle u, w\rangle \perp W,\ V = \langle v, w\rangle \perp W.$$

Suppose $f(u, v) = 0$ holds. Then since $f(u+v, w) = 1+1 = 0$ and $f(u+v, v) = 0$, we have $u + v \in \mathbb{V}^{\perp} = \langle r\rangle$ so that $v = u + \alpha r$ for some $\alpha \in \mathbb{F}_q$. This implies $u = v$, since $0 = Q(v) = \alpha^2$, which contradicts the assumption $U \neq V$. Therefore we may assume $f(u, v) = 1$ without loss of generality, since $Q(w) = 0$.

Let U' and V' be other distinct elements in Ω such that $U' \cap V'$ is degenerate, and decompose U' and V' in the same manner:

$$U' = \langle u', w'\rangle \perp W', V' = \langle v', w'\rangle \perp W',$$

where $\{u', w'\}, \{v', w'\}$ are hyperbolic pairs, W' is a positive-type hyperplane of $U' \cap V'$, and $f(u', v') = 1$. Let $\tau : W \longrightarrow W'$ be an isometry, and define a linear mapping $\tilde{\tau} : \mathbb{V} \longrightarrow \mathbb{V}$ by $\tilde{\tau}|_W := \tau$, $\tilde{\tau}(u) := u'$, $\tilde{\tau}(v) := v'$, and $\tilde{\tau}(w) := w'$. Then $\tilde{\tau}$ becomes an automorphism of Q and we have $\tau(U) = U'$, $\tau(V) = V'$. Hence it follows that

$$R_1 := \{(U, V) \in \Omega \times \Omega \mid U \cap V :\ \text{degenerate}\} \qquad (4)$$

forms a relation of $\mathfrak{X}(GO_{2m+1}(q),\Omega)$ (that is, an orbital of the transitive action of $GO_{2m+1}(q)$ on Ω).

Finally we determine the valency k_1 of R_1. Let H be a degenerate hyperplane of U, then any non-degenerate hyperplane K of $\mathbb{V}$ which includes H becomes automatically positive-type. In fact, since there exist a singular vector w in H and a positive-type hyperplane W of H such that

$$H = \langle w\rangle \perp W,$$

hence $K = W \perp (W^{\perp} \cap K)$ cannot be negative-type. There are $\frac{q^{2m+1}-q^{2m-1}}{q^{2m}-q^{2m-1}} = q+1$ hyperplanes of $\mathbb{V}$ which include H. In these $q+1$ hyperplanes, $\langle r\rangle \perp H$ is the only degenerate hyperplane. Thus there are $q-1$ elements V in Ω such that $(U, V) \in R_1$ and $U \cap V = H$. It follows from Proposition 1.1.1 that there is a one-to-one correspondence between degenerate hyperplanes of U and 1-dimensional singular subspaces of U. Therefore by Lemma 1.1.3(ii) we have

$$k_1 = \frac{q^m + q^{m-1}(q^m-1) - 1}{q-1}(q-1) = (q^{m-1}+1)(q^m-1). \qquad (5)$$

(ii) Suppose $U \cap V$ is a non-degenerate subspace in $\mathbb{V}$. Then there exists a vector w in $U \cap V$ such that $Q(w) = 1$ and

$$U \cap V = \langle w \rangle^{\perp} \cap U = \langle w \rangle^{\perp} \cap V.$$

First of all, we show that for any non-degenerate hyperplane W of $U \cap V$ there exist two vectors $u \in U$, $v \in V$ such that $Q(u) = Q(v)$, $f(u,w) = f(v,w) = 1$ and

$$U = \langle u, w \rangle \perp W, \ V = \langle v, w \rangle \perp W.$$

If W is positive-type (resp. negative-type), then $W^{\perp} \cap U$ is also positive-type (resp. negative-type). Let $u \in W^{\perp} \cap U$ and $v \in W^{\perp} \cap V$ be two vectors such that $f(u,w) = 1$ and $f(v,w) = 1$, then the polynomials $t^2 + t + Q(u)$ and $t^2 + t + Q(v)$ are reducible (resp. irreducible) over $\mathbb{F}_q$. The assertion follows immediately from the fact that the set $\{\alpha^2 + \alpha \,|\, \alpha \in \mathbb{F}_q\}$ is an additive subgroup of $\mathbb{F}_q$ of index 2 (cf. Munemasa [12, p.12, Lemma 2.9.]). In fact, let α be an element in $\mathbb{F}_q$ such that $Q(u) = \alpha^2 + \alpha + Q(v)$ then $u \in U$ and $v' := \alpha w + v \in V$ are desired vectors, since $Q(v') = \alpha^2 + \alpha + Q(v) = Q(u)$ and $f(v', w) = f(v, w) = 1$.

Define

$$\Delta := \frac{f(u,v)}{f(u,v)+1}. \tag{6}$$

Then we have the following:

Proposition 1.2.1. *Δ is well-defined and $\Delta \in \mathbb{F}_q \backslash \{0,1\}$. Moreover, the pair $\{\Delta, \Delta^{-1}\}$ does not depend on the choice of W, u, v.*

Proof. Since $f(u+v, w) = 1 + 1 = 0$, the vector $u + v$ is contained in $(U \cap V)^{\perp}$. By Proposition 1.1.1 we have $\dim(U \cap V)^{\perp} = 2$, from which it follows that

$$u + v = \alpha w + \beta r$$

for some α, $\beta \in \mathbb{F}_q$.

Suppose $f(u,v) = 0$, that is $\alpha = 0$. Then we have $\beta = 0$, since $Q(u) = Q(v) = Q(u + \beta r) = Q(u) + \beta^2$. This implies $u = v$, which is a contradiction. Next, suppose $f(u,v) = 1$, that is $\alpha = 1$. Then we also have $\beta = 0$, since $Q(u) = Q(v) = Q(u + w + \beta r) = Q(u) + 1 + 1 + \beta^2 = Q(u) + \beta^2$. In this case this implies $u + w = v$, which is also a contradiction.

In order to show that the pair $\{\Delta, \Delta^{-1}\}$ does not depend on W, u and v, let

$$U = \langle u', w \rangle \perp W', \ V = \langle v', w \rangle \perp W'$$

be another decomposition such that $Q(u') = Q(v')$ and $f(u', w) = f(v', w) = 1$. Then since $f(u', w) = 1$, we have $u' = u + \gamma w + z$ for some $\gamma \in \mathbb{F}_q$ and $z \in W$. Let $v'' := v + \gamma w + z$ be a vector in V, then clearly $Q(v'') = Q(u')$ and $f(v'', w) = 1$. Furthermore we have $u' + v'' = u + v = \alpha w + \beta r$, that is,

$$V = \langle v'', w \rangle \perp W',$$

which implies that v'' must be v' or $v' + w$, since if we express v'' as a linear combination of v' and w, say $v'' = \gamma_1 v' + \gamma_2 w$, then $\gamma_1 = f(v'', w) = 1$ and $Q(v') = Q(v'') = Q(v') + \gamma_2^2 + \gamma_2$ so that $\gamma_2 = 0$ or $\gamma_2 = 1$. If $v'' = v'$, then we have

$$\frac{f(u',v')}{f(u',v')+1} = \frac{f(u',v'')}{f(u',v'')+1} = \frac{f(u,v)}{f(u,v)+1}.$$

Similarly if $v'' = v' + w$, then we have

$$\frac{f(u',v')}{f(u',v')+1} = \frac{f(u',v''+w)}{f(u',v''+w)+1} = \frac{f(u',v'')+1}{f(u',v')} = \frac{f(u,v)+1}{f(u,v)}.$$

This completes the proof of Proposition 1.2.1. $\square$

We denote

$$\Delta(U,V) := \{\Delta, \Delta^{-1}\}. \tag{7}$$

It should be noticed that in the definition of $\Delta(U,V)$ it does not matter whether W is positive-type or negative-type.

Let U' and V' be other distinct two elements in Ω such that $\Delta(U',V') = \Delta(U,V)$, and let

$$U' = \langle u', w'\rangle \perp W', \ V' = \langle v', w'\rangle \perp W'$$

be a decomposition, where $Q(w') = 1$, W' has the same type as W, $Q(u') = Q(v') = Q(u)$ and $f(u',w') = f(v',w') = 1$. Without loss of generality we may assume $f(u,v) = f(u',v')$. Let $\tau : W \longrightarrow W'$ be an isometry, and define a linear mapping $\tilde{\tau} : \mathbb{V} \longrightarrow \mathbb{V}$ by $\tilde{\tau}|_W := \tau$, $\tilde{\tau}(u) := u'$, $\tilde{\tau}(v) := v'$, and $\tilde{\tau}(w) := w'$. Then $\tilde{\tau}$ becomes an automorphism of Q and we have $\tau(U) = U'$, $\tau(V) = V'$. Thus, the remaining relations of $\mathfrak{X}(GO_{2m+1}(q), \Omega)$ are described as follows:

$$R_i := \left\{(U,V) \in \Omega \times \Omega \;\middle|\; U \cap V : \text{non-degenerate}, \ \Delta(U,V) = \{\nu^{i-1}, \nu^{-(i-1)}\}\right\} \quad (2 \le i \le \tfrac{q}{2}), \tag{8}$$

where $\nu \in \mathbb{F}_q^*$ is a primitive element of $\mathbb{F}_q$.

Finally, we determine the valencies k_i of R_i $(2 \le i \le \frac{q}{2})$. We define

$$\lambda_i := \frac{\nu^{i-1}}{1+\nu^{i-1}} \in \mathbb{F}_q \backslash \{0,1\}, \quad \text{for } 2 \le i \le \tfrac{q}{2}, \tag{9}$$

and

$$\mu_i := \sqrt{\lambda_i^2 + \lambda_i} \neq 0, \quad \text{for } 2 \le i \le \tfrac{q}{2}, \tag{10}$$

Notice that

$$\frac{\nu^{-(i-1)}}{1+\nu^{-(i-1)}} = \frac{1}{1+\nu^{i-1}} = \lambda_i + 1,$$

from which it follows that

$$\lambda_i + \lambda_j \neq 0, 1, \quad \text{if } i \neq j, \tag{11}$$

or equivalently

$$\mu_i \neq \mu_j \quad \text{if } i \neq j. \tag{12}$$

Let H be a non-degenerate hyperplane of U, then there exists a vector w in H such that $Q(w) = 1$ and

$$H = \langle w \rangle^{\perp} \cap U.$$

Fix a vector u in U such that $f(u,w) = 1$. Then it follows that the only element V of Ω which satisfies $U \cap V = H$ and $\Delta(U,V) = \{\nu^{i-1}, \nu^{-(i-1)}\}$ is given by

$$V := \langle v \rangle \oplus H, \text{ where } v := u + \lambda_i w + \mu_i r \in \mathbb{V}. \tag{13}$$

To show this, let V be such an element in Ω and let

$$U = \langle u, w\rangle \perp W, \ V = \langle v, w \rangle \perp W$$

be a decomposition, where $Q(v) = Q(u)$ and $f(v,w) = 1$. As is in the proof of Proposition 1.2.1, $u + v = \alpha w + \beta r$ for some $\alpha, \beta \in \mathbb{F}_q$, where $\alpha \neq 0, 1$, and we may assume

$$\frac{f(u,v)}{f(u,v)+1} = \nu^{i-1}$$

without loss of generality. Then we have

$$\nu^{i-1} = \frac{\alpha}{\alpha+1},$$

from which it follows $\alpha = \lambda_i$. Also we have $Q(u) = Q(v) = Q(u) + \lambda_i^2 + \lambda_i + \beta^2$ so that $\beta = \mu_i$, as desired.

It follows from Proposition 1.1.1 that there is a one-to-one correspondence between non-degenerate hyperplanes of U and 1-dimensional non-singular subspaces of U. Therefore by Lemma 1.1.3(ii) we obtain

$$k_i = q^{m-1}(q^m - 1) \tag{14}$$

for $2 \le i \le \frac{q}{2}$. To summarize:

The association scheme $\mathfrak{X}(GO_{2m+1}, \Omega) = (\Omega, \{R_i\}_{0 \le i \le \frac{q}{2}})$ is a symmetric association scheme of class $\frac{q}{2}$ whose relations are defined by

$$R_1 := \{(U,V) \in \Omega \times \Omega \mid U \cap V : \text{ degenerate}\},$$
$$R_i := \big\{(U,V) \in \Omega \times \Omega \;\big|\; U \cap V : \text{non-degenerate, } \Delta(U,V) = \{\nu^{i-1}, \nu^{-(i-1)}\}\big\} \quad (2 \le i \le \tfrac{q}{2}).$$

The valencies of $\mathfrak{X}(GO_{2m+1}, \Omega)$ are given as

$$k_1 = (q^{m-1} + 1)(q^m - 1),$$
$$k_i = q^{m-1}(q^m - 1) \quad (2 \le i \le \tfrac{q}{2}).$$

Secondly, we describe the relations of the association scheme $\mathfrak{X}(GO_{2m+1}(q), \Theta)$ in the same way as $\mathfrak{X}(GO_{2m+1}, \Omega)$. Let U, V be two distinct elements in Θ.

(i) Suppose $U \cap V$ is a degenerate subspace in $\mathbb{V}$. Notice that this occurs only if $m \ge 2$, since any 2-dimensional negative-type subspace of $\mathbb{V}$ has no non-zero singular vector. There exists a singular vector w in $U \cap V$ such that

$$U \cap V = \langle w \rangle^\perp \cap U = \langle w \rangle^\perp \cap V,$$

Let u be a vector in U such that $\{u, w\}$ is a hyperbolic pair (Proposition 1.1.2). Then since U and V are both negative-type, there exists a negative-type hyperplane W of $U \cap V$ and a vector $v \in V$ such that $\{v, w\}$ is a hyperbolic pair and

$$U = \langle u, w \rangle \perp W, \; V = \langle v, w \rangle \perp W.$$

Suppose $f(u, v) = 0$ holds. Then since $f(u + v, w) = 1 + 1 = 0$ and $f(u + v, v) = 0$, we have $u + v \in \mathbb{V}^\perp = \langle r \rangle$ so that $v = u + \alpha r$ for some $\alpha \in \mathbb{F}_q$. This implies $u = v$, since $0 = Q(v) = \alpha^2$, which contradicts the assumption $U \ne V$. Therefore we may assume $f(u, v) = 1$ without loss of generality, since $Q(w) = 0$.

Let U' and V' be other distinct elements in Θ such that $U' \cap V'$ is degenerate, and decompose U' and V' in the same manner:

$$U' = \langle u', w' \rangle \perp W', V' = \langle v', w' \rangle \perp W',$$

where $\{u', w'\}, \{v', w'\}$ are hyperbolic pairs, W' is a negative-type hyperplane of $U' \cap V'$, and $f(u', v') = 1$. Let $\tau : W \longrightarrow W'$ be an isometry, and define a linear mapping $\tilde{\tau} : \mathbb{V} \longrightarrow \mathbb{V}$ by $\tilde{\tau}|_W := \tau$, $\tilde{\tau}(u) := u'$, $\tilde{\tau}(v) := v'$, and $\tilde{\tau}(w) := w'$. Then $\tilde{\tau}$ becomes an automorphism of Q and we have $\tau(U) = U'$, $\tau(V) = V'$. Hence it follows that

$$S_1 := \{(U,V) \in \Theta \times \Theta \mid U \cap V : \text{ degenerate}\} \tag{15}$$

forms a relation of $\mathfrak{X}(GO_{2m+1}(q), \Theta)$.

Finally we determine the valency h_1 of S_1. Let H be a degenerate hyperplane of U, then any non-degenerate hyperplane K of $\mathbb{V}$ which includes H becomes automatically negative-type. In fact, since there exist a singular vector w in H and a negative-type hyperplane W of H such that

$$H = \langle w \rangle \perp W,$$

hence $K = W \perp (W^\perp \cap K)$ cannot be positive-type. There are $\frac{q^{2m+1} - q^{2m-1}}{q^{2m} - q^{2m-1}} = q+1$ hyperplanes of $\mathbb{V}$ which include H. In these $q+1$ hyperplanes, $\langle r \rangle \perp H$ is the only degenerate hyperplane. Thus there are $q-1$ elements V in Θ such that $(U,V)\in S_1$ and $U\cap V=H$. It follows from Proposition 1.1.1 that there is a one-to-one correspondence between degenerate hyperplanes of U and 1-dimensional singular subspaces of U. Therefore by Lemma 1.1.3(iii) we have

$$h_1=\frac{q^{m-1}+q^m(q^{m-1}-1)-1}{q-1}(q-1)=(q^{m-1}-1)(q^m+1). \tag{16}$$

(ii) Suppose $U\cap V$ is a non-degenerate subspace in $\mathbb{V}$. Then there exists a vector w in $U\cap V$ such that $Q(w)=1$ and

$$U\cap V=\langle w\rangle^{\perp}\cap U=\langle w\rangle^{\perp}\cap V.$$

First of all, we show that for any non-degenerate hyperplane W of $U\cap V$ there exist two vectors $u\in U$, $v\in V$ such that $Q(u)=Q(v)$, $f(u,w)=f(v,w)=1$ and

$$U=\langle u,w\rangle\perp W,\ V=\langle v,w\rangle\perp W.$$

If W is positive-type (resp. negative-type), then $W^{\perp}\cap U$ is negative-type (resp. positive-type). Let $u\in W^{\perp}\cap U$ and $v\in W^{\perp}\cap V$ be two vectors such that $f(u,w)=1$ and $f(v,w)=1$, then the polynomials $t^2+t+Q(u)$ and $t^2+t+Q(v)$ are irreducible (resp. reducible) over $\mathbb{F}_q$. The assertion follows immediately from the fact that the set $\{\alpha^2+\alpha\,|\,\alpha\in\mathbb{F}_q\}$ is an additive subgroup of $\mathbb{F}_q$ of index 2 (cf. Munemasa [12, p.12, Lemma 2.9.]). In fact, let α be an element in $\mathbb{F}_q$ such that $Q(u)=\alpha^2+\alpha+Q(v)$ then $u\in U$ and $v':=\alpha w+v\in V$ are desired vectors, since $Q(v')=\alpha^2+\alpha+Q(v)=Q(u)$ and $f(v',w)=f(v,w)=1$.

Define

$$\Pi:=\frac{f(u,v)}{f(u,v)+1}. \tag{17}$$

The proof of the following proposition is exactly the same as that of Proposition 1.2.1.

Proposition 1.2.2. Π *is well-defined and* $\Pi\in\mathbb{F}_q\backslash\{0,1\}$. *Moreover, the pair* $\{\Pi,\Pi^{-1}\}$ *does not depend on the choice of* W,u,v.

Proof. Since $f(u+v,w)=1+1=0$, the vector $u+v$ is contained in $(U\cap V)^{\perp}$. By Proposition 1.1.1 we have $\dim(U\cap V)^{\perp}=2$, from which it follows that $u+v=\alpha w+\beta r$ for some $\alpha,\ \beta\in\mathbb{F}_q$.

Suppose $f(u,v)=0$, that is $\alpha=0$. Then we have $\beta=0$, since $Q(u)=Q(v)=Q(u+\beta r)=Q(u)+\beta^2$. This implies $u=v$, which is a contradiction. Next, suppose $f(u,v)=1$, that is $\alpha=1$. Then we also have $\beta=0$, since $Q(u)=Q(v)=Q(u+w+\beta r)=Q(u)+1+1+\beta^2=Q(u)+\beta^2$. In this case this implies $u+w=v$, which is also a contradiction.

In order to show that the pair $\{\Pi,\Pi^{-1}\}$ does not depend on W,u and v, let

$$U=\langle u',w\rangle\perp W',\ V=\langle v',w\rangle\perp W'$$

be another decomposition such that $Q(u')=Q(v')$ and $f(u',w)=f(v',w)=1$. Then since $f(u',w)=1$, we have $u'=u+\gamma w+z$ for some $\gamma\in\mathbb{F}_q$ and $z\in W$. Let $v'':=v+\gamma w+z$ be a vector in V, then clearly $Q(v'')=Q(u')$ and $f(v'',w)=1$. Furthermore we have $u'+v''=u+v=\alpha w+\beta r$, that is,

$$V=\langle v'',w\rangle\perp W',$$

which implies that v'' must be v' or $v'+w$, since if we express v'' as a linear combination of v' and w, say $v''=\gamma_1 v'+\gamma_2 w$, then $\gamma_1=f(v'',w)=1$ and $Q(v')=Q(v'')=Q(v')+\gamma_2^2+\gamma_2$ so that $\gamma_2=0$ or $\gamma_2=1$. If $v''=v'$, then we have

$$\frac{f(u',v')}{f(u',v')+1}=\frac{f(u',v'')}{f(u',v'')+1}=\frac{f(u,v)}{f(u,v)+1}.$$

Similarly if $v''=v'+w$, then we have

$$\frac{f(u',v')}{f(u',v')+1}=\frac{f(u',v''+w)}{f(u',v''+w)+1}=\frac{f(u',v'')+1}{f(u',v')}=\frac{f(u,v)+1}{f(u,v)}.$$

This completes the proof of Proposition 1.2.2. $\square$

We denote

$$\Pi(U,V) := \{\Pi, \Pi^{-1}\}. \tag{18}$$

It should be noticed that in the definition of $\Pi(U,V)$ it does not matter whether W is positive-type or negative-type.

Let U' and V' be other distinct two elements in Θ such that $\Pi(U',V') = \Pi(U,V)$, and let

$$U' = \langle u', w'\rangle \perp W',\ V' = \langle v', w'\rangle \perp W'$$

be a decomposition, where $Q(w') = 1$, W' has the same type as W, $Q(u') = Q(v') = Q(u)$ and $f(u',w') = f(v',w') = 1$. Without loss of generality we may assume $f(u,v) = f(u',v')$. Let $\tau : W \longrightarrow W'$ be an isometry, and define a linear mapping $\tilde{\tau} : \mathbb{V} \longrightarrow \mathbb{V}$ by $\tilde{\tau}|_W := \tau$, $\tilde{\tau}(u) := u'$, $\tilde{\tau}(v) := v'$, and $\tilde{\tau}(w) := w'$. Then $\tilde{\tau}$ becomes an automorphism of Q and we have $\tau(U) = U'$, $\tau(V) = V'$. Thus, the remaining relations of $\mathfrak{X}(GO_{2m+1}(q),\Theta)$ are described as follows:

$$S_i := \big\{(U,V) \in \Theta\times\Theta \bigm| U\cap V : \text{non-degenerate},\ \Pi(U,V) = \{\nu^{i-1}, \nu^{-(i-1)}\}\big\} \quad (2 \le i \le \tfrac{q}{2}). \tag{19}$$

Finally, we determine the valencies h_i of S_i $(2 \le i \le \frac{q}{2})$. Let H be a non-degenerate hyperplane of U, then there exists a vector w in H such that $Q(w) = 1$ and

$$H = \langle w\rangle^{\perp} \cap U.$$

Fix a vector u in U such that $f(u,w) = 1$. Then it follows that the only element V of Θ which satisfies $U \cap V = H$ and $\Pi(U,V) = \{\nu^{i-1}, \nu^{-(i-1)}\}$ is given by

$$V := \langle v\rangle \oplus H, \text{ where } v := u + \lambda_i w + \mu_i r \in \mathbb{V}. \tag{20}$$

To show this, let V be such an element in Θ and let

$$U = \langle u, w\rangle \perp W,\ V = \langle v, w\rangle \perp W$$

be a decomposition, where $Q(v) = Q(u)$ and $f(v,w) = 1$. As is in the proof of Proposition 1.2.2, $u + v = \alpha w + \beta r$ for some $\alpha, \beta \in \mathbb{F}_q$, where $\alpha \neq 0, 1$, and we may assume

$$\frac{f(u,v)}{f(u,v)+1} = \nu^{i-1}$$

without loss of generality. Then we have

$$\nu^{i-1} = \frac{\alpha}{\alpha+1},$$

from which it follows $\alpha = \lambda_i$. Also we have $Q(u) = Q(v) = Q(u) + \lambda_i^2 + \lambda_i + \beta^2$ so that $\beta = \mu_i$, as desired.

It follows from Proposition 1.1.1 that there is a one-to-one correspondence between non-degenerate hyperplanes of U and 1-dimensional non-singular subspaces of U. Therefore by Lemma 1.1.3(iii) we obtain

$$h_i = q^{m-1}(q^m + 1) \tag{21}$$

for $2 \le i \le \frac{q}{2}$. To summarize:

The association scheme $\mathfrak{X}(GO_{2m+1},\Theta) = (\Theta, \{S_i\}_{0\le i\le \frac{q}{2}})$ is a symmetric association scheme of class $\frac{q}{2}$ if $m \ge 2$, whose relations are defined by

$$S_1 := \{(U,V) \in \Theta\times\Theta \mid U\cap V : \text{ degenerate}\},$$
$$S_i := \big\{(U,V) \in \Theta\times\Theta \bigm| U\cap V : \text{non-degenerate},\ \Pi(U,V) = \{\nu^{i-1}, \nu^{-(i-1)}\}\big\} \quad (2 \le i \le \tfrac{q}{2}).$$

If $m = 1$ then we have $S_1 = \emptyset$ so that $\mathfrak{X}(GO_3(q),\Theta)$ is a symmetric association scheme of class $\frac{q}{2} - 1$. The valencies of $\mathfrak{X}(GO_{2m+1},\Theta)$ are given as

$$h_1 = (q^{m-1} - 1)(q^m + 1),$$
$$h_i = q^{m-1}(q^m + 1) \quad (2 \le i \le \tfrac{q}{2}).$$

2 Computation of Parameters

2.1 The Parameters of $\mathfrak{X}(GO_{2m+1}(q), \Omega_{2m+1}(q))$

In this subsection, we compute the intersection numbers $\{p_{ij}^k\}$ of $\mathfrak{X}(GO_{2m+1}(q), \Omega_{2m+1}(q))$.

(i) Suppose first $2 \leq i, j, k \leq \frac{q}{2}$. Let U and V be elements in Ω such that $(U, V) \in R_k$, and let w denote the vector in $H := U \cap V$ such that $Q(w) = 1$ and

$$H = U \cap V = \langle w \rangle^{\perp} \cap U = \langle w \rangle^{\perp} \cap V.$$

First of all, we count the number of elements K in Ω which satisfy $(U, K) \in R_i$, $(V, K) \in R_j$, and $U \cap K = V \cap K = H$. Let v be a vector in V with $f(v, w) = 1$ and define

$$u := v + \lambda_k w + \mu_k r.$$

Then it follows from (13) that

$$U = H \oplus \langle u \rangle,$$

and the only element K in Ω such that $(V, K) \in R_j$ and $V \cap K = H$ is given by

$$K := H \oplus \langle z \rangle, \quad \text{where } z := v + \lambda_j w + \mu_j r.$$

Since

$$u + z = (\lambda_j + \lambda_k)w + (\mu_j + \mu_k)r,$$

if $(U, K) \in R_i$, then we have

$$\frac{\lambda_j + \lambda_k}{\lambda_j + \lambda_k + 1} = \nu^{\pm(i-1)},$$

that is,

$$\lambda_j + \lambda_k = \lambda_i, \quad \text{or} \quad \lambda_j + \lambda_k = \lambda_i + 1,$$

which is equivalent to

$$\mu_i^2 + \mu_j^2 + \mu_k^2 = 0.$$

Thus the number n_1 of elements K in Ω which satisfy $(U, K) \in R_i$, $(V, K) \in R_j$, and $U \cap K = V \cap K = H$ is

$$n_1 = \begin{cases} 1 & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 = 0, \\ 0 & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 \neq 0. \end{cases} \tag{22}$$

Next, fix a vector w' in $V \backslash \langle w \rangle$ with $Q(w') = 1$, and define two subspaces of V as

$$\begin{aligned} H' :=& \langle w' \rangle^{\perp} \cap V, \\ W :=& H \cap H' = \langle w, w' \rangle^{\perp} \cap V. \end{aligned}$$

We need to determine whether there exists an element K in Ω such that $(U, K) \in R_i$, $(V, K) \in R_j$ and $V \cap K = H'$. Notice that $\operatorname{Rad} f|_{\langle w, w' \rangle} = 0$ holds if and only if $f(w, w') \neq 0$, that is, W is non-degenerate if and only if $f(w, w') \neq 0$. Assume $f(w, w') \neq 0$, then W is positive-type if and only if the polynomial $t^2 + f(w, w')t + 1 \in \mathbb{F}_q[t]$ is reducible over $\mathbb{F}_q$, since a 2-dimensional negative-type subspace has no non-zero singular vector.

Let v be a vector in H such that $f(v, w') = 1$, and let v' be a vector in H' such that $f(v', w) = 1$. We define

$$\begin{aligned} u :=& v' + \lambda_k w + \mu_k r, \\ z :=& v + \lambda_j w' + \mu_j r, \end{aligned}$$

so that $U = H \oplus \langle u \rangle$, and $K := H' \oplus \langle z \rangle$ is the unique element in Ω which satisfies $(V, K) \in R_j$ and $V \cap K = H'$.

Since $\mathbb{V} = W \oplus \langle v, v' \rangle \oplus \langle r \rangle$, any vector x in $U \cap K$ is uniquely written as

$$x = \alpha v + \beta v' + y + \gamma r,$$

for some $\alpha, \beta, \gamma \in \mathbb{F}_q$ and $y \in W$. Then it follows from $U = H \oplus \langle u \rangle$ and $K = H' \oplus \langle z \rangle$ that

$$\gamma = \alpha\mu_j = \beta\mu_k. \tag{23}$$

Notice that there exist two vectors y and y' in W such that

$$w = f(w, w')v + y, \quad w' = f(w, w')v' + y'. \tag{24}$$

Let w'' be a vector in $U \cap K$ such that

$$U \cap K = \langle w'' \rangle^{\perp} \cap U = \langle w'' \rangle^{\perp} \cap K.$$

(a) Suppose $f(w, w') \neq 0$, that is, W is a non-degenerate subspace of V. Since $\dim W^{\perp} = 3$ by Proposition 1.1.1, there exist some elements $\xi, \eta, \delta \in \mathbb{F}_q$ such that

$$w'' = \xi w + \eta w' + \delta r.$$

Then by (24) we have

$$w'' = \xi f(w, w')v + \eta f(w, w')v' + \xi y + \eta y' + \delta r,$$

so that from (23) we obtain

$$w'' = \mu_k \epsilon w + \mu_j \epsilon w' + \mu_j \mu_k f(w, w')\epsilon r, \tag{25}$$

for some $\epsilon \in \mathbb{F}_q$. Since $w'' \neq 0$, we have $\epsilon \neq 0$.

Now suppose $(U, K) \in R_i$, then $Q(w'')$ must not be 0. Hence the inner product $f(w, w')$ must satisfy

$$\mu_j^2 \mu_k^2 f(w, w')^2 + \mu_j \mu_k f(w, w') + \mu_j^2 + \mu_k^2 \neq 0,$$

or equivalently

$$f(w, w') \neq \frac{1}{\mu_j \mu_k}(\lambda_j + \lambda_k),\ \frac{1}{\mu_j \mu_k}(\lambda_j + \lambda_k + 1).$$

We may assume $Q(w'') = 1$ so that

$$\epsilon^2 = \frac{1}{\mu_j^2 \mu_k^2 f(w, w')^2 + \mu_j \mu_k f(w, w') + \mu_j^2 + \mu_k^2}. \tag{26}$$

If $(U, K) \in R_i$ then repeating the same argument as before we have

$$w = \mu_i \epsilon' w'' + \mu_j \epsilon' w' + \mu_i \mu_j f(w'', w')\epsilon' r, \tag{27}$$

for some $\epsilon' \in \mathbb{F}_q$. Then since it follows from (25) that

$$w = \frac{1}{\mu_k \epsilon} w'' + \frac{\mu_j}{\mu_k} w' + \mu_j f(w, w') r,$$

we have

$$\epsilon = \frac{1}{\mu_i}.$$

Therefore by (26) the inner product $f(w, w')$ must satisfy the following condition:

$$\mu_j^2 \mu_k^2 f(w, w')^2 + \mu_j \mu_k f(w, w') + \mu_i^2 + \mu_j^2 + \mu_k^2 = 0,$$

which is equivalent to

$$f(w, w') = \frac{1}{\mu_j \mu_k}(\lambda_i + \lambda_j + \lambda_k), \text{ or } f(w, w') = \frac{1}{\mu_j \mu_k}(\lambda_i + \lambda_j + \lambda_k + 1). \tag{28}$$

Conversely, if $f(w, w')$ satisfies (28) then from (11) we deduce that $(U, K) \in R_i$.

We can now count the number of elements K in Ω such that $(U, K) \in R_i$, $(V, K) \in R_j$, and $W = U \cap V \cap K = H \cap K$ is a non-degenerate hyperplane of H. For brevity we let

$$\kappa_{ijk} := \frac{1}{\mu_j \mu_k}(\lambda_i + \lambda_j + \lambda_k)$$
$$\kappa'_{ijk} := \frac{1}{\mu_j \mu_k}(\lambda_i + \lambda_j + \lambda_k + 1)$$

for $i, j, k \in \{2, 3, \ldots, \frac{q}{2}\}$, and define

$$\phi(\alpha) := \begin{cases} 1 & \text{if the polynomial } t^2 + \alpha t + 1 \in \mathbb{F}_q[t] \text{ is reducible over } \mathbb{F}_q, \\ 0 & \text{otherwise,} \end{cases}$$

for $\alpha \in \mathbb{F}_q$. (For $\alpha \in \mathbb{F}_q^*$ the function $\phi(\alpha)$ is also defined by $\phi(\alpha) = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\alpha^{-1})$. See Lemma 4.2.4 below.) We need the following lemma:

Lemma 2.1.1. *Let W_0 be a 2-dimensional positive-type (resp. negative-type) subspace of $\mathbb{V}$, and let w be a vector in W_0 with $Q(w) = 1$. Then for any $\alpha \in \mathbb{F}_q^*$ such that $\phi(\alpha) = 1$ (resp. $\phi(\alpha) = 0$) there exist two vectors w'_1, w'_2 in W_0 such that $Q(w'_i) = 1$ and $f(w, w'_i) = \alpha$ $(i = 1, 2)$.*

Proof. Let y be a vector in $W_0 \backslash \langle w \rangle$ with $Q(y) = 1$, then we have $f(w, y) \neq 0$ and $\phi(f(w, y)) = 1$ (resp. $\phi(f(w, y)) = 0$). If a vector y' in W_0 satisfies $Q(y') = 1$ and $f(w, y') = f(w, y)$, then y' must be y or $y + f(w, y)w$. The number of elements $\alpha \in \mathbb{F}_q^*$ such that $\phi(\alpha) = 1$ (resp. $\phi(\alpha) = 0$) is obviously equal to $\frac{q}{2} - 1$ (resp. $\frac{q}{2}$), and by Proposition 1.1.3 the number of vectors y in W_0 other than w with $Q(y) = 1$ is given by $q - 2$ (resp. q), which proves the lemma. □

Suppose for instance $\kappa_{ijk} \neq 0$ and $\phi(\kappa_{ijk}) = 1$ (resp. $\phi(\kappa_{ijk}) = 0$). If $w' \in V$ satisfies $Q(w') = 1$ and $f(w, w') = \kappa_{ijk}$, then as mentioned before, $W := \langle w, w' \rangle^\perp \cap V$ is a positive-type (resp. negative-type) hyperplane of H. On the other hand, let W be a positive-type (resp. negative-type) hyperplane of H, then by Lemma 2.1.1 the number of vectors w' in $W^\perp \cap V$ which satisfy $Q(w') = 1$ and $f(w, w') = \kappa_{ijk}$ is exactly 2. Thus from (2) (resp. (3)), the number of vectors w' in V such that $Q(w') = 1$ and $f(w, w') = \kappa_{ijk}$ is given by

$$q^{m-1}(q^{m-1} + 1) \quad \big(\text{resp. } q^{m-1}(q^{m-1} - 1)\big).$$

Since κ_{ijk}, $\kappa'_{ijk} \neq 0$ unless $\mu_i^2 + \mu_j^2 + \mu_k^2 = 0$, the number n_2 of elements K in Ω such that $(U, K) \in R_i$, $(V, K) \in R_j$ and $W = U \cap V \cap K = H \cap K$ is a non-degenerate hyperplane of H, is obtained as follows:

$$n_2 = \begin{cases} q^{m-1}(q^{m-1} + 1) & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 = 0 \text{ and } \phi\left(\frac{1}{\mu_j \mu_k}\right) = 1, \\ q^{m-1}(q^{m-1} - 1) & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 = 0 \text{ and } \phi\left(\frac{1}{\mu_j \mu_k}\right) = 0, \\ 2q^{m-1}(q^{m-1} + 1) & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 \neq 0 \text{ and } \phi(\kappa_{ijk}) = \phi(\kappa'_{ijk}) = 1, \\ 2q^{m-1}(q^{m-1} - 1) & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 \neq 0 \text{ and } \phi(\kappa_{ijk}) = \phi(\kappa'_{ijk}) = 0, \\ 2q^{2m-2} & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 \neq 0 \text{ and } \{\phi(\kappa_{ijk}), \phi(\kappa'_{ijk})\} = \{0, 1\}. \end{cases} \tag{29}$$

(b) Suppose $f(w, w') = 0$, that is, W is a degenerate subspace of V. Notice that this occurs only if $m \geq 2$. In this case $\langle w, w' \rangle$ is a subspace of W. Since $\dim W^\perp \cap K = 2$ by Proposition 1.1.1 we have $W^\perp \cap K = \langle w, w' \rangle$. Therefore there exist two elements ξ and η in $\mathbb{F}_q$ such that

$$w'' = \xi w + \eta w'.$$

Let $x = \alpha v + \beta v' + y + \gamma r$ be a vector in $U \cap K$, then it follows from (23) that

$$0 = \mu_k f(x, w'') = \beta \mu_k \xi + \alpha \mu_k \eta = \alpha(\mu_j \xi + \mu_k \eta),$$

so that

$$w'' = \mu_k \epsilon w + \mu_j \epsilon w', \tag{30}$$

for some $\epsilon \in \mathbb{F}_q$.

Now suppose $(U, K) \in R_i$, then $Q(w'')$ must not be 0, that is, $\mu_j^2 + \mu_k^2 \neq 0$. We may assume $Q(w'') = 1$ so that

$$\epsilon^2 = \frac{1}{\mu_j^2 + \mu_k^2}.$$

If $(U, K) \in R_i$ then repeating the same argument as before we have

$$w = \mu_i \epsilon' w'' + \mu_j \epsilon' w', \tag{31}$$

for some $\epsilon' \in \mathbb{F}_q$. Then since it follows from (30) that

$$w = \frac{1}{\mu_k \epsilon} w'' + \frac{\mu_j}{\mu_k} w',$$

we have

$$\epsilon = \frac{1}{\mu_i},$$

so that

$$\mu_i^2 + \mu_j^2 + \mu_k^2 = 0. \tag{32}$$

Conversely, if (32) is satisfied, then from (12) we deduce $(U, K) \in R_i$. By Lemma 1.1.3(i) there are $q^{2m-2} - 1$ vectors w' in H other than w such that $Q(w') = 1$, hence the number n_3 of elements K in Ω such that $(U, K) \in R_i$, $(V, K) \in R_j$ and $W = U \cap V \cap K = H \cap K$ is a degenerate hyperplane of H, is given by

$$n_3 = \begin{cases} q^{2m-2} - 1 & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 = 0, \\ 0 & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 \neq 0. \end{cases} \tag{33}$$

From (22), (29) and (33), we obtain

$$\begin{aligned} p_{ij}^k &= n_1 + n_2 + n_3 \\ &= \begin{cases} q^{m-1}(2q^{m-1} + 1) & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 = 0 \text{ and } \phi\left(\frac{1}{\mu_j \mu_k}\right) = 1, \\ q^{m-1}(2q^{m-1} - 1) & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 = 0 \text{ and } \phi\left(\frac{1}{\mu_j \mu_k}\right) = 0, \\ 2q^{m-1}(q^{m-1} + 1) & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 \neq 0 \text{ and } \phi(\kappa_{ijk}) = \phi(\kappa'_{ijk}) = 1, \\ 2q^{m-1}(q^{m-1} - 1) & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 \neq 0 \text{ and } \phi(\kappa_{ijk}) = \phi(\kappa'_{ijk}) = 0, \\ 2q^{2m-2} & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 \neq 0 \text{ and } \{\phi(\kappa_{ijk}), \phi(\kappa'_{ijk})\} = \{0, 1\}, \end{cases} \end{aligned} \tag{34}$$

for $i, j, k \in \{2, 3, \ldots, \frac{q}{2}\}$.

(ii) Suppose $i = 1$ and $2 \leq j, k \leq \frac{q}{2}$. Let U and V be elements in Ω such that $(U, V) \in R_k$. We use the same notation as in (i). Notice that if an element K in Ω satisfies $(U, K) \in R_1$ and $(V, K) \in R_j$, then $U \cap V \cap K = H \cap K$ has dimension $2m - 2$, since $U \cap K$ is degenerate by definition while H is non-degenerate.

In the same way as (i), fix a vector w' in $V \backslash \langle w \rangle$ with $Q(w') = 1$, and define two subspaces of V as

$$\begin{aligned} H' &:= \langle w' \rangle^{\perp} \cap V, \\ W &:= H \cap H' = \langle w, w' \rangle^{\perp} \cap V. \end{aligned}$$

Let K be the unique element in Ω which satisfies $(V, K) \in R_j$ and $V \cap K = H'$, and let w'' be a vector in $U \cap K$ such that $U \cap K = \langle w'' \rangle^{\perp} \cap U = \langle w'' \rangle^{\perp} \cap K$. Then $(U, K) \in R_1$ if and only if $Q(w'') = 0$.

(a) Suppose $f(w, w') \neq 0$, that is, W is a non-degenerate subspace of V, then it follows from (25) that $(U, K) \in R_1$ if and only if

$$\mu_j^2 \mu_k^2 f(w, w')^2 + \mu_j \mu_k f(w, w') + \mu_j^2 + \mu_k^2 = 0,$$

which is equivalent to

$$f(w,w') = \frac{1}{\mu_j\mu_k}(\lambda_j+\lambda_k), \text{ or } f(w,w') = \frac{1}{\mu_j\mu_k}(\lambda_j+\lambda_k+1). \tag{35}$$

It follows that

$$\phi\left(\frac{1}{\mu_j\mu_k}(\lambda_j+\lambda_k)\right) = \phi\left(\frac{1}{\mu_j\mu_k}(\lambda_j+\lambda_k+1)\right) = 1. \tag{36}$$

In fact,

$$\begin{aligned}
\frac{1}{\mu_j^2\mu_k^2}(\lambda_j^2+\lambda_k^2) &= \frac{1}{(\lambda_j^2+\lambda_j)(\lambda_k^2+\lambda_k)}(\lambda_j^2+\lambda_k^2)\\
&= \frac{1}{(\lambda_j^2+\lambda_j)(\lambda_k^2+\lambda_k)}(\lambda_j^2(\lambda_k^2+1)+\lambda_k^2(\lambda_j^2+1))\\
&= \Big(\frac{\lambda_j}{\lambda_j+1}\Big)\Big(\frac{\lambda_k+1}{\lambda_k}\Big) + \Big(\frac{\lambda_k}{\lambda_k+1}\Big)\Big(\frac{\lambda_j+1}{\lambda_j}\Big)\\
&= \nu^{j-k}+\nu^{-(j-k)}.
\end{aligned}$$

Likewise

$$\begin{aligned}
\frac{1}{\mu_j^2\mu_k^2}(\lambda_j^2+\lambda_k^2+1) &= \frac{1}{(\lambda_j^2+\lambda_j)(\lambda_k^2+\lambda_k)}(\lambda_j^2+\lambda_k^2+1)\\
&= \frac{1}{(\lambda_j^2+\lambda_j)(\lambda_k^2+\lambda_k)}(\lambda_j^2\lambda_k^2+(\lambda_j^2+1)(\lambda_k^2+1))\\
&= \Big(\frac{\lambda_j}{\lambda_j+1}\Big)\Big(\frac{\lambda_k}{\lambda_k+1}\Big) + \Big(\frac{\lambda_j+1}{\lambda_j}\Big)\Big(\frac{\lambda_k+1}{\lambda_k}\Big)\\
&= \nu^{j+k}+\nu^{-(j+k)}.
\end{aligned}$$

It follows from (2) that the number of positive-type hyperplanes of H is given by

$$\frac{q^{m-1}(q^{m-1}+1)}{2}.$$

Hence by Lemma 2.1.1 the number n'_2 of elements K in Ω such that $(U,K)\in R_1$, $(V,K)\in R_j$ and $W=U\cap V\cap K=H\cap K$ is a non-degenerate hyperplane of H, is obtained as

$$n'_2 = \begin{cases} q^{m-1}(q^{m-1}+1) & \text{if } j=k,\\ 2q^{m-1}(q^{m-1}+1) & \text{if } j\neq k,\end{cases} \tag{37}$$

since n'_2 is equal to the number of vectors w' in $V\backslash H$ with $Q(w')=1$ which satisfies (35).

(b) Suppose $f(w,w')=0$, that is, W is a degenerate subspace of V, which occurs only if $m\geq 2$. Then it follows from (30) that $(U,K)\in R_1$ if and only if

$$\mu_j^2+\mu_k^2=0,$$

that is, $j=k$. By Lemma 1.1.3(i), the number of vectors w' in H other than w with $Q(w')=1$ is equal to $q^{2m-2}-1$, from which it follows that the number n'_3 of elements K in Ω such that $(U,K)\in R_1$, $(V,K)\in R_j$ and $W=U\cap V\cap K=H\cap K$ is a degenerate hyperplane of H, is given by

$$n'_3 = \begin{cases} q^{2m-2}-1 & \text{if } j=k,\\ 0 & \text{if } j\neq k.\end{cases} \tag{38}$$

From (37) and (38), we obtain

$$p_{1j}^k = n'_2+n'_3 = \begin{cases} (2q^{m-1}-1)(q^{m-1}+1) & \text{if } j=k,\\ 2q^{m-1}(q^{m-1}+1) & \text{if } j\neq k.\end{cases} \tag{39}$$

(iii) Suppose $2 \leq i \leq \frac{q}{2}$ and $j = k = 1$. Let U and V be elements in Ω such that $(U, V) \in R_1$, and let w denote a vector in $H := U \cap V$ such that $Q(w) = 0$ and

$$H = U \cap V = \langle w \rangle^{\perp} \cap U = \langle w \rangle^{\perp} \cap V.$$

Notice that in this case w is not uniquely determined, and also notice that if an element K in Ω satisfies $(U, K) \in R_i$ and $(V, K) \in R_1$, then $U \cap V \cap K = H \cap K$ has dimension $2m-2$, since $U \cap K$ is non-degenerate by definition while H is degenerate. Fix a vector w' in $V \backslash \langle w \rangle$ with $Q(w') = 0$, and define two subspaces of V as

$$\begin{aligned} H' :=& \langle w' \rangle^{\perp} \cap V, \\ W :=& H \cap H' = \langle w, w' \rangle^{\perp} \cap V. \end{aligned}$$

We determine whether there exists an element K in Ω such that $(U, K) \in R_i$, $(V, K) \in R_1$ and $V \cap K = H'$.

(a) Suppose $f(w, w') \neq 0$, that is, W is a non-degenerate hyperplane of H. Since $Q(w) = Q(w') = 0$, we may assume $f(w, w') = 1$ without loss of generality. Define

$$v_\alpha := w + w' + \alpha r,$$

for $\alpha \in \mathbb{F}_q^*$, then $K_\alpha := H' \oplus \langle v_\alpha \rangle$ ($\alpha \in \mathbb{F}_q^*$) are distinct elements in Ω with $K_\alpha \cap V = H'$. In fact, since these hyperplanes of $\mathbb{V}$ do not contain the vector r, they are non-degenerate. Moreover since W is positive-type and $H' = W \perp \langle w' \rangle$, we conclude that $K_\alpha = W \perp (W^\perp \cap K_\alpha)$ is positive-type for all $\alpha \in \mathbb{F}_q^*$, and also it follows that they are distinct elements in Ω since we have $v_\alpha + v_\beta = (\alpha + \beta) r$ for $\alpha, \beta \in \mathbb{F}_q^*$. The number of hyperplanes K of $\mathbb{V}$ which include H' is given by $\frac{q^{2m+1} - q^{2m-1}}{q^{2m} - q^{2m-1}} = q+1$. In these $q+1$ hyperplanes of $\mathbb{V}$, $H' \perp \langle r \rangle$ is the only degenerate hyperplane, that is, there are $q-1$ elements K in Ω such that $K \cap V = H'$ and hence each K is written as $K = K_\alpha$ for some $\alpha \in \mathbb{F}_q^*$. By the same reason, there exists an element α_0 in $\mathbb{F}_q^*$ such that

$$U = H \oplus \langle v_{\alpha_0} \rangle.$$

Fix an element α in $\mathbb{F}_q^*$, then since $\mathbb{V} = W \oplus \langle w, w' \rangle \oplus \langle r \rangle$, any vector x in $U \cap K_\alpha$ is uniquely written as

$$x = \xi w + \eta w' + y + \delta r,$$

for some $\xi, \eta, \delta \in \mathbb{F}_q$ and $y \in W$. Then it follows from $U = H \oplus \langle v_{\alpha_0} \rangle$ and $K_\alpha = H' \oplus \langle v_\alpha \rangle$ that

$$\delta = \xi \alpha = \eta \alpha_0. \tag{40}$$

Let w'' be a vector in $U \cap K_\alpha$ such that

$$U \cap K_\alpha = \langle w'' \rangle^{\perp} \cap U = \langle w'' \rangle^{\perp} \cap K_\alpha.$$

Since $W^\perp = \langle w, w', r \rangle$, it follows from (40) that

$$w'' = \alpha_0 \epsilon w + \alpha \epsilon w' + \alpha_0 \alpha \epsilon r, \tag{41}$$

for an element $\epsilon \in \mathbb{F}_q$. Since $w'' \neq 0$, we have $\epsilon \neq 0$.

Now suppose $(U, K_\alpha) \in R_i$, then $Q(w'')$ must not be 0, so that

$$\alpha_0^2 \alpha^2 + \alpha_0 \alpha \neq 0,$$

which is equivalent to

$$\alpha \neq \alpha_0^{-1}.$$

We may assume $Q(w'') = 1$ so that

$$\epsilon^2 = \frac{1}{\alpha_0^2 \alpha^2 + \alpha_0 \alpha}.$$

Then we have

$$\frac{1}{\alpha\epsilon}w + \frac{1}{\alpha_0\epsilon}w' = \frac{1}{\alpha_0\alpha\epsilon^2}w'' + \frac{1}{\epsilon}r = (\alpha_0\alpha + 1)w'' + \frac{1}{\epsilon}r. \tag{42}$$

From $f(w, w'') = \alpha\epsilon$, $f(w', w'') = \alpha_0\epsilon$ and $Q(w) = Q(w') = 0$ it follows that

$$\frac{\alpha_0\alpha + 1}{\alpha_0\alpha} = \nu^{i-1} \quad \text{or} \quad \frac{\alpha_0\alpha + 1}{\alpha_0\alpha} = \nu^{-(i-1)},$$

or equivalently

$$\alpha_0\alpha = \lambda_i \quad \text{or} \quad \alpha_0\alpha = \lambda_i + 1. \tag{43}$$

Conversely if (43) is satisfied, then from (11) we deduce that $(U, K) \in R_i$. Therefore for each 1-dimensional singular subspace $\langle w' \rangle$ in V such that $f(w, w') \neq 0$, there are exactly 2 elements K in Ω such that $(U, K) \in R_i$, $(V, K) \in R_1$ and $V \cap K = H' := \langle w' \rangle^\perp \cap V$. The number of vectors v in V such that $f(v, w) \neq 0$ is $q^{2m} - q^{2m-1}$, and hence the number of 2-dimensional positive-type subspaces of V which include $\langle w \rangle$ is by Proposition 1.1.2 equal to

$$\frac{q^{2m} - q^{2m-1}}{q^2 - q} = q^{2m-2}, \tag{44}$$

which is also equal to the number of 1-dimensional singular subspace $\langle w' \rangle$ in V such that $f(w, w') \neq 0$ since any 2-dimensional positive-type subspace of $\mathbb{V}$ contains two 1-dimensional singular subspaces. Thus the number m_1 of elements K in Ω such that $(U, K) \in R_i$, $(V, K) \in R_1$ and $W = U \cap V \cap K = H \cap K$ is a non-degenerate hyperplane of H, is given by

$$m_1 = 2q^{2m-2}. \tag{45}$$

(b) Suppose $f(w, w') = 0$, which never happens if $m = 1$, then $\langle w, w' \rangle = W^\perp \cap V$ is a singular subspace of W. Hence if an element K in Ω satisfies $U \cap V \cap K = W$, then $U \cap K$ cannot be non-degenerate, since $(U \cap K)^\perp \cap K \subset W^\perp \cap K = \langle w, w' \rangle$. This implies that there is no element K in Ω such that $(U, K) \in R_i$, $(V, K) \in R_1$ and $W = U \cap V \cap K = H \cap K$ is a degenerate hyperplane of H.

Thus by (45)

$$p^1_{i1} = m_1 = 2q^{2m-2}. \tag{46}$$

(iv) Finally suppose $i = j = k = 1$. Let U and V be elements in Ω such that $(U, V) \in R_1$. We use the same notation as in (iii). In the same way as (iii), fix a vector w' in $V \backslash \langle w \rangle$ with $Q(w') = 1$, and define two subspaces of V as

$$\begin{aligned} H' :=& \langle w' \rangle^\perp \cap V, \\ W :=& H \cap H' = \langle w, w' \rangle^\perp \cap V. \end{aligned}$$

(a) Suppose $f(w, w') \neq 0$, that is, W is a non-degenerate hyperplane of H. We may assume $f(w, w') = 1$ without loss of generality, since $Q(w) = Q(w') = 0$. Define

$$v_\alpha := w + w' + \alpha r,$$

for $\alpha \in \mathbb{F}_q^*$, then as mentioned before, $U = H \oplus \langle v_{\alpha_0} \rangle$ for some $\alpha_0 \in \mathbb{F}_q^*$. Also $K_\alpha := H' \oplus \langle v_\alpha \rangle$ ($\alpha \in \mathbb{F}_q^*$) are distinct elements in Ω with $K_\alpha \cap V = H'$, and each element K in Ω such that $K \cap V = H'$ is written as $K = K_\alpha$ for some $\alpha \in \mathbb{F}_q^*$.

Fix an element α in $\mathbb{F}_q^*$ and let w'' be a vector in $U \cap K_\alpha$ such that $U \cap K_\alpha = \langle w'' \rangle^\perp \cap U = \langle w'' \rangle^\perp \cap K_\alpha$. Then $(U, K) \in R_1$ if and only if $Q(w'') = 0$, which is by (41) equivalent to

$$\alpha_0^2\alpha^2 + \alpha_0\alpha = 0,$$

that is, $\alpha = \alpha_0^{-1}$. Therefore for each 1-dimensional singular subspace $\langle w' \rangle$ in V such that $f(w, w') \neq 0$, there is exactly one element K in Ω such that $(U, K) \in R_1$, $(V, K) \in R_1$ and $V \cap K = H' := \langle w' \rangle^{\perp} \cap V$. Hence it follows from (44) that the number m_1' of elements K in Ω such that $(U, K) \in R_1$, $(V, K) \in R_1$ and $W = U \cap V \cap K = H \cap K$ is a non-degenerate hyperplane of H, is given by

$$m_1' = q^{2m-2}. \tag{47}$$

(b) Suppose $f(w, w') = 0$, that is, W is a degenerate hyperplane of H. This happens only if $m \geq 2$. In this case any element K in Ω such that $U \cap V \cap K = W$ satisfies $(U, K) \in R_1$, since $(U \cap K)^{\perp} \cap K \subset W^{\perp} \cap K = \langle w, w' \rangle$ and $\langle w, w' \rangle$ is a singular subspace.

The number of singular vectors in H is given by

$$q^m + q^{m-1}(q^{m-1} - 1). \tag{48}$$

To show this, let W' be a non-degenerate hyperplane of H so that we have $H = W' \perp \langle w \rangle$. Since W' is positive-type and $Q(w) = 0$, it follows from Lemma 1.1.3(ii) that the number of singular vectors in H is equal to

$$q\{q^{m-1} + q^{m-2}(q^{m-1} - 1)\} = q^m + q^{m-1}(q^{m-1} - 1),$$

as desired. Thus by (48) the number m_2' of elements K in Ω such that $(U, K) \in R_1$, $(V, K) \in R_1$ and $W = U \cap V \cap K = H \cap K$ is a degenerate hyperplane of H, is given by

$$m_2' = \frac{q^m + q^{m-1}(q^{m-1} - 1) - q}{q - 1}(q - 1) = q^m + q^{m-1}(q^{m-1} - 1) - q. \tag{49}$$

(c) We have to count the number m_3' of the elements K in Ω such that $(U, K) \in R_1$, $(V, K) \in R_1$ and $U \cap K = H$. Since there are exactly $\frac{q^{2m+1} - q^{2m-1}}{q^{2m} - q^{2m-1}} - 1 = q$ elements in Ω which include H, m_3' is given by

$$m_3' = q - 2. \tag{50}$$

From (47), (49) and (50) we obtain

$$p_{11}^1 = m_1' + m_2' + m_3' = q^{m-1}(2q^{m-1} + q - 1) - 2. \tag{51}$$

The rest of parameters are directly computed by the following equality (cf. Bannai-Ito [4, p.55, Proposition 2.2.]):

Proposition 2.1.2. *Let $\{p_{ij}^k\}$ denotes the intersection numbers of a symmetric association scheme $\mathfrak{X} = (X, \{R_i\}_{0 \leq i \leq d})$. Then for all $i, j, k \in \{0, 1, \ldots, \frac{q}{2}\}$ we have $k_k p_{ij}^k = k_j p_{ki}^j = k_i p_{jk}^i$.*

Hence from (46)

$$p_{11}^k = \frac{k_1}{k_k} p_{k1}^1 = 2q^{m-1}(q^{m-1} + 1) \quad \text{for } 2 \leq k \leq \tfrac{q}{2}. \tag{52}$$

Also from (39)

$$p_{ij}^1 = \frac{k_j}{k_1} p_{1i}^j = \begin{cases} q^{m-1}(2q^{m-1} - 1) & \text{if } 2 \leq i = j \leq \frac{q}{2}, \\ 2q^{2m-2} & \text{if } 2 \leq i, j \leq \frac{q}{2} \text{ and } i \neq j. \end{cases} \tag{53}$$

To summarize:

Lemma 2.1.3. *The intersection numbers p_{ij}^k of $\mathfrak{X}(GO_{2m+1}(q), \Omega)$ are given as follows.*

$$p_{ij}^1 = p_{ji}^1 = \begin{cases} q^{m-1}(2q^{m-1}+q-1)-2 & \text{if } i=j=1, \\ q^{m-1}(2q^{m-1}-1) & \text{if } 2 \leq i = j \leq \frac{q}{2}, \\ 2q^{2m-2} & \text{if } 1 \leq i < j \leq \frac{q}{2}, \end{cases}$$

$$p_{1j}^k = p_{j1}^k = \begin{cases} (2q^{m-1}-1)(q^{m-1}+1) & \text{if } 2 \leq j = k \leq \frac{q}{2}, \\ 2q^{m-1}(q^{m-1}+1) & \text{if } 1 \leq j \leq \frac{q}{2},\ 2 \leq k \leq \frac{q}{2},\ j \neq k. \end{cases}$$

For other $2 \leq i, j, k \leq \frac{q}{2}$,

$$p_{ij}^k = \begin{cases} q^{m-1}(2q^{m-1}+1) & \text{if } \mu_i^2+\mu_j^2+\mu_k^2 = 0 \text{ and } \phi\left(\frac{1}{\mu_j\mu_k}\right) = 1, \\ q^{m-1}(2q^{m-1}-1) & \text{if } \mu_i^2+\mu_j^2+\mu_k^2 = 0 \text{ and } \phi\left(\frac{1}{\mu_j\mu_k}\right) = 0, \\ 2q^{m-1}(q^{m-1}+1) & \text{if } \mu_i^2+\mu_j^2+\mu_k^2 \neq 0 \text{ and } \phi(\kappa_{ijk}) = \phi(\kappa'_{ijk}) = 1, \\ 2q^{m-1}(q^{m-1}-1) & \text{if } \mu_i^2+\mu_j^2+\mu_k^2 \neq 0 \text{ and } \phi(\kappa_{ijk}) = \phi(\kappa'_{ijk}) = 0, \\ 2q^{2m-2} & \text{if } \mu_i^2+\mu_j^2+\mu_k^2 \neq 0 \text{ and } \{\phi(\kappa_{ijk}), \phi(\kappa'_{ijk})\} = \{0,1\}. \end{cases}$$

2.2 The Parameters of $\mathfrak{X}(GO_{2m+1}(q), \Theta_{2m+1}(q))$

In this subsection, we compute the intersection numbers $\{s_{ij}^k\}$ of $\mathfrak{X}(GO_{2m+1}(q), \Theta_{2m+1}(q))$.

(i) Suppose first $2 \leq i, j, k \leq \frac{q}{2}$. Let U and V be elements in Θ such that $(U, V) \in S_k$, and let w denote the vector in $H := U \cap V$ such that $Q(w) = 1$ and

$$H = U \cap V = \langle w \rangle^\perp \cap U = \langle w \rangle^\perp \cap V.$$

First of all, we count the number of elements K in Θ which satisfy $(U, K) \in S_i$, $(V, K) \in S_j$, and $U \cap K = V \cap K = H$. Let v be a vector in V with $f(v, w) = 1$ and define

$$u := v + \lambda_k w + \mu_k r.$$

Then it follows from (20) that

$$U = H \oplus \langle u \rangle,$$

and the only element K in Θ such that $(V, K) \in S_j$ and $V \cap K = H$ is given by

$$K := H \oplus \langle z \rangle, \quad \text{where } z := v + \lambda_j w + \mu_j r.$$

Since

$$u + z = (\lambda_j + \lambda_k)w + (\mu_j + \mu_k)r,$$

if $(U, K) \in S_i$, then we have

$$\frac{\lambda_j + \lambda_k}{\lambda_j + \lambda_k + 1} = \nu^{\pm(i-1)},$$

that is,

$$\lambda_j + \lambda_k = \lambda_i, \quad \text{or} \quad \lambda_j + \lambda_k = \lambda_i + 1,$$

which is equivalent to

$$\mu_i^2 + \mu_j^2 + \mu_k^2 = 0.$$

Thus the number n_1 of elements K in Θ which satisfy $(U, K) \in S_i$, $(V, K) \in S_j$, and $U \cap K = V \cap K = H$ is

$$n_1 = \begin{cases} 1 & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 = 0, \\ 0 & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 \neq 0. \end{cases} \tag{54}$$

Next, fix a vector w' in $V \backslash \langle w \rangle$ with $Q(w') = 1$, and define two subspaces of V as

$$\begin{aligned} H' &:= \langle w' \rangle^\perp \cap V, \\ W &:= H \cap H' = \langle w, w' \rangle^\perp \cap V. \end{aligned}$$

We need to determine whether there exists an element K in Θ such that $(U,K) \in S_i$, $(V,K) \in S_j$ and $V \cap K = H'$. Notice that $\mathrm{Rad}\, f|_{\langle w,w'\rangle} = 0$ holds if and only if $f(w,w') \neq 0$, that is, W is non-degenerate if and only if $f(w,w') \neq 0$. Assume $f(w,w') \neq 0$, then W is negative-type if and only if the polynomial $t^2 + f(w,w')t + 1 \in \mathbb{F}_q[t]$ is reducible over $\mathbb{F}_q$, since a 2-dimensional negative-type subspace has no non-zero singular vector.

Let v be a vector in H such that $f(v,w') = 1$, and let v' be a vector in H' such that $f(v',w) = 1$. We define

$$u := v' + \lambda_k w + \mu_k r,$$
$$z := v + \lambda_j w' + \mu_j r,$$

so that $U = H \oplus \langle u \rangle$, and $K := H' \oplus \langle z \rangle$ is the unique element in Θ which satisfies $(V,K) \in S_j$ and $V \cap K = H'$.

Since $\mathbb{V} = W \oplus \langle v, v' \rangle \oplus \langle r \rangle$, any vector x in $U \cap K$ is uniquely written as

$$x = \alpha v + \beta v' + y + \gamma r,$$

for some $\alpha, \beta, \gamma \in \mathbb{F}_q$ and $y \in W$. Then it follows from $U = H \oplus \langle u \rangle$ and $K = H' \oplus \langle z \rangle$ that

$$\gamma = \alpha\mu_j = \beta\mu_k. \tag{55}$$

Notice that there exist two vectors y and y' in W such that

$$w = f(w,w')v + y, \quad w' = f(w,w')v' + y'. \tag{56}$$

Let w'' be a vector in $U \cap K$ such that

$$U \cap K = \langle w'' \rangle^{\perp} \cap U = \langle w'' \rangle^{\perp} \cap K.$$

(a) Suppose $f(w,w') \neq 0$, that is, W is a non-degenerate subspace of V. Since by Proposition 1.1.1 we have $\dim W^{\perp} = 3$, there exist some elements $\xi, \eta, \delta \in \mathbb{F}_q$ such that

$$w'' = \xi w + \eta w' + \delta r.$$

Then by (56) we have

$$w'' = \xi f(w,w')v + \eta f(w,w')v' + \xi y + \eta y' + \delta r,$$

so that from (55) we obtain

$$w'' = \mu_k \epsilon w + \mu_j \epsilon w' + \mu_j \mu_k f(w,w')\epsilon r, \tag{57}$$

for some $\epsilon \in \mathbb{F}_q$. Since $w'' \neq 0$, we have $\epsilon \neq 0$.

Now suppose $(U,K) \in S_i$, then $Q(w'')$ must not be 0. Hence the inner product $f(w,w')$ must satisfy

$$\mu_j^2 \mu_k^2 f(w,w')^2 + \mu_j \mu_k f(w,w') + \mu_j^2 + \mu_k^2 \neq 0,$$

or equivalently

$$f(w,w') \neq \frac{1}{\mu_j \mu_k}(\lambda_j + \lambda_k),\ \frac{1}{\mu_j \mu_k}(\lambda_j + \lambda_k + 1).$$

We may assume $Q(w'') = 1$ so that

$$\epsilon^2 = \frac{1}{\mu_j^2 \mu_k^2 f(w,w')^2 + \mu_j \mu_k f(w,w') + \mu_j^2 + \mu_k^2}. \tag{58}$$

If $(U,K) \in S_i$ then repeating the same argument as before we have

$$w = \mu_i \epsilon' w'' + \mu_j \epsilon' w' + \mu_i \mu_j f(w'',w')\epsilon' r, \tag{59}$$

for some $\epsilon' \in \mathbb{F}_q$. Then since it follows from (57) that

$$w = \frac{1}{\mu_k \epsilon} w'' + \frac{\mu_j}{\mu_k} w' + \mu_j f(w, w') r,$$

we have

$$\epsilon = \frac{1}{\mu_i}.$$

Therefore by (58) the inner product $f(w, w')$ must satisfy the following condition:

$$\mu_j^2 \mu_k^2 f(w, w')^2 + \mu_j \mu_k f(w, w') + \mu_i^2 + \mu_j^2 + \mu_k^2 = 0,$$

which is equivalent to

$$f(w, w') = \frac{1}{\mu_j \mu_k} (\lambda_i + \lambda_j + \lambda_k), \text{ or } f(w, w') = \frac{1}{\mu_j \mu_k} (\lambda_i + \lambda_j + \lambda_k + 1). \tag{60}$$

Conversely, if $f(w, w')$ satisfies (60) then from (11) we deduce that $(U, K) \in S_i$.

We can now count the number of elements K in Θ such that $(U, K) \in S_i$, $(V, K) \in S_j$, and $W = U \cap V \cap K = H \cap K$ is a non-degenerate hyperplane of H. We use the same notation as the previous subsection.

Suppose for instance $\kappa_{ijk} \neq 0$ and $\phi(\kappa_{ijk}) = 1$ (resp. $\phi(\kappa_{ijk}) = 0$). If $w' \in V$ satisfies $Q(w') = 1$ and $f(w, w') = \kappa_{ijk}$, then as mentioned before, $W := \langle w, w' \rangle^\perp \cap V$ is a negative-type (resp. positive-type) hyperplane of H. On the other hand, let W be a negative-type (resp. positive-type) hyperplane of H, then by Lemma 2.1.1 the number of vectors w' in $W^\perp \cap V$ which satisfy $Q(w') = 1$ and $f(w, w') = \kappa_{ijk}$ is exactly 2. Thus from (3) (resp. (2)), the number of vectors w' in V such that $Q(w') = 1$ and $f(w, w') = \kappa_{ijk}$ is given by

$$q^{m-1}(q^{m-1} - 1) \quad \left(\text{resp. } q^{m-1}(q^{m-1} + 1)\right).$$

Since κ_{ijk}, $\kappa'_{ijk} \neq 0$ unless $\mu_i^2 + \mu_j^2 + \mu_k^2 = 0$, the number n_2 of elements K in Θ such that $(U, K) \in S_i$, $(V, K) \in S_j$ and $W = U \cap V \cap K = H \cap K$ is a non-degenerate hyperplane of H, is obtained as follows:

$$n_2 = \begin{cases} q^{m-1}(q^{m-1} - 1) & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 = 0 \text{ and } \phi\left(\frac{1}{\mu_j \mu_k}\right) = 1, \\ q^{m-1}(q^{m-1} + 1) & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 = 0 \text{ and } \phi\left(\frac{1}{\mu_j \mu_k}\right) = 0, \\ 2q^{m-1}(q^{m-1} - 1) & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 \neq 0 \text{ and } \phi(\kappa_{ijk}) = \phi(\kappa'_{ijk}) = 1, \\ 2q^{m-1}(q^{m-1} + 1) & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 \neq 0 \text{ and } \phi(\kappa_{ijk}) = \phi(\kappa'_{ijk}) = 0, \\ 2q^{2m-2} & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 \neq 0 \text{ and } \{\phi(\kappa_{ijk}), \phi(\kappa'_{ijk})\} = \{0, 1\}. \end{cases} \tag{61}$$

(b) Suppose $f(w, w') = 0$, that is, W is a degenerate subspace of V. Notice that this occurs only if $m \geq 2$. In this case $\langle w, w' \rangle$ is a subspace of W. Since $\dim W^\perp \cap K = 2$ by Proposition 1.1.1 we have $W^\perp \cap K = \langle w, w' \rangle$. Therefore there exist two elements ξ and η in $\mathbb{F}_q$ such that

$$w'' = \xi w + \eta w'.$$

Let $x = \alpha v + \beta v' + y + \gamma r$ be a vector in $U \cap K$, then it follows from (55) that

$$0 = \mu_k f(x, w'') = \beta \mu_k \xi + \alpha \mu_k \eta = \alpha(\mu_j \xi + \mu_k \eta),$$

so that

$$w'' = \mu_k \epsilon w + \mu_j \epsilon w', \tag{62}$$

for some $\epsilon \in \mathbb{F}_q$.

Now suppose $(U, K) \in S_i$, then $Q(w'')$ must not be 0, that is, $\mu_j^2 + \mu_k^2 \neq 0$. We may assume $Q(w'') = 1$ so that

$$\epsilon^2 = \frac{1}{\mu_j^2 + \mu_k^2}.$$

If $(U,K) \in S_i$ then repeating the same argument as before we have

$$w = \mu_i \epsilon' w'' + \mu_j \epsilon' w', \tag{63}$$

for some $\epsilon' \in \mathbb{F}_q$. Then since it follows from (62) that

$$w = \frac{1}{\mu_k \epsilon} w'' + \frac{\mu_j}{\mu_k} w',$$

we have

$$\epsilon = \frac{1}{\mu_i},$$

so that

$$\mu_i^2 + \mu_j^2 + \mu_k^2 = 0. \tag{64}$$

Conversely, if (64) is satisfied, then from (12) we deduce $(U,K) \in S_i$. By Lemma 1.1.3(i) there are $q^{2m-2} - 1$ vectors w' in H other than w such that $Q(w') = 1$, hence the number n_3 of elements K in Θ such that $(U,K) \in S_i$, $(V,K) \in S_j$ and $W = U \cap V \cap K = H \cap K$ is a degenerate hyperplane of H, is given by

$$n_3 = \begin{cases} q^{2m-2} - 1 & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 = 0, \\ 0 & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 \neq 0. \end{cases} \tag{65}$$

From (54), (61) and (65), we obtain

$$\begin{aligned} s_{ij}^k &= n_1 + n_2 + n_3 \\ &= \begin{cases} q^{m-1}(2q^{m-1} - 1) & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 = 0 \text{ and } \phi\left(\frac{1}{\mu_j \mu_k}\right) = 1, \\ q^{m-1}(2q^{m-1} + 1) & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 = 0 \text{ and } \phi\left(\frac{1}{\mu_j \mu_k}\right) = 0, \\ 2q^{m-1}(q^{m-1} - 1) & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 \neq 0 \text{ and } \phi(\kappa_{ijk}) = \phi(\kappa'_{ijk}) = 1, \\ 2q^{m-1}(q^{m-1} + 1) & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 \neq 0 \text{ and } \phi(\kappa_{ijk}) = \phi(\kappa'_{ijk}) = 0, \\ 2q^{2m-2} & \text{if } \mu_i^2 + \mu_j^2 + \mu_k^2 \neq 0 \text{ and } \{\phi(\kappa_{ijk}), \phi(\kappa'_{ijk})\} = \{0, 1\}, \end{cases} \end{aligned} \tag{66}$$

for $i, j, k \in \{2, 3, \ldots, \frac{q}{2}\}$.

Notice that we have computed all the intersection numbers of $\mathfrak{X}(GO_3(q), \Theta_3(q))$, since if $m = 1$ then $S_1 = \emptyset$. Thus in what follows, we always assume $m \geq 2$.

(ii) Suppose $i = 1$ and $2 \leq j, k \leq \frac{q}{2}$. Let U and V be elements in Θ such that $(U,V) \in S_k$. We use the same notation as in (i). Notice that if an element K in Θ satisfies $(U,K) \in S_1$ and $(V,K) \in S_j$, then $U \cap V \cap K = H \cap K$ has dimension $2m-2$, since $U \cap K$ is degenerate by definition while H is non-degenerate.

In the same way as (i), fix a vector w' in $V \backslash \langle w \rangle$ with $Q(w') = 1$, and define two subspaces of V as

$$\begin{aligned} H' &:= \langle w' \rangle^\perp \cap V, \\ W &:= H \cap H' = \langle w, w' \rangle^\perp \cap V. \end{aligned}$$

Let K be the unique element in Θ which satisfies $(V,K) \in S_j$ and $V \cap K = H'$, and let w'' be a vector in $U \cap K$ such that $U \cap K = \langle w'' \rangle^\perp \cap U = \langle w'' \rangle^\perp \cap K$. Then $(U,K) \in S_1$ if and only if $Q(w'') = 0$.

(a) Suppose $f(w,w') \neq 0$, that is, W is a non-degenerate subspace of V, then it follows from (57) that $(U,K) \in S_1$ if and only if

$$\mu_j^2 \mu_k^2 f(w,w')^2 + \mu_j \mu_k f(w,w') + \mu_j^2 + \mu_k^2 = 0,$$

which is equivalent to

$$f(w,w') = \frac{1}{\mu_j \mu_k}(\lambda_j + \lambda_k), \text{ or } f(w,w') = \frac{1}{\mu_j \mu_k}(\lambda_j + \lambda_k + 1), \tag{67}$$

where as shown in the previous subsection (cf. (36)) we have

$$\phi\left(\frac{1}{\mu_j\mu_k}(\lambda_j+\lambda_k)\right)=\phi\left(\frac{1}{\mu_j\mu_k}(\lambda_j+\lambda_k+1)\right)=1.$$

It follows from (3) that the number of negative-type hyperplanes of H is given by

$$\frac{q^{m-1}(q^{m-1}-1)}{2}.$$

Hence by Lemma 2.1.1 the number n_2' of elements K in Θ such that $(U,K)\in S_1$, $(V,K)\in S_j$ and $W=U\cap V\cap K=H\cap K$ is a non-degenerate hyperplane of H, is obtained as

$$n_2'=\begin{cases}q^{m-1}(q^{m-1}-1) & \text{if } j=k,\\ 2q^{m-1}(q^{m-1}-1) & \text{if } j\neq k,\end{cases} \tag{68}$$

since n_2' is equal to the number of vectors w' in $V\backslash H$ with $Q(w')=1$ which satisfies (67).

(b) Suppose $f(w,w')=0$, that is, W is a degenerate subspace of V. Then it follows from (62) that $(U,K)\in S_1$ if and only if

$$\mu_j^2+\mu_k^2=0,$$

that is, $j=k$. By Lemma 1.1.3(i), the number of vectors w' in H other than w with $Q(w')=1$ is equal to $q^{2m-2}-1$, from which it follows that the number n_3' of elements K in Θ such that $(U,K)\in S_1$, $(V,K)\in S_j$ and $W=U\cap V\cap K=H\cap K$ is a degenerate hyperplane of H, is given by

$$n_3'=\begin{cases}q^{2m-2}-1 & \text{if } j=k,\\ 0 & \text{if } j\neq k.\end{cases} \tag{69}$$

From (68) and (69), we obtain

$$s_{1j}^k=n_2'+n_3'=\begin{cases}(2q^{m-1}+1)(q^{m-1}-1) & \text{if } j=k,\\ 2q^{m-1}(q^{m-1}-1) & \text{if } j\neq k.\end{cases} \tag{70}$$

(iii) Suppose $2\le i\le \frac{q}{2}$ and $j=k=1$. Let U and V be elements in Θ such that $(U,V)\in S_1$, and let w denote a vector in $H:=U\cap V$ such that $Q(w)=0$ and

$$U\cap V=\langle w\rangle^\perp\cap U=\langle w\rangle^\perp\cap V.$$

Notice that in this case w is not uniquely determined, and also notice that if an element K in Θ satisfies $(U,K)\in S_i$ and $(V,K)\in S_1$, then $U\cap V\cap K=H\cap K$ has dimension $2m-2$, since $U\cap K$ is non-degenerate by definition while H is degenerate. Fix a vector w' in $V\backslash\langle w\rangle$ with $Q(w')=0$, and define two subspaces of V as

$$\begin{aligned}H' :=&\langle w'\rangle^\perp\cap V,\\ W :=&H\cap H'=\langle w,w'\rangle^\perp\cap V.\end{aligned}$$

We determine whether there exists an element K in Θ such that $(U,K)\in S_i$, $(V,K)\in S_1$ and $V\cap K=H'$.

(a) Suppose $f(w,w')\neq 0$, that is, W is a non-degenerate hyperplane of H. Since $Q(w)=Q(w')=0$, we may assume $f(w,w')=1$ without loss of generality. Define

$$v_\alpha:=w+w'+\alpha r,$$

for $\alpha\in\mathbb{F}_q^*$, then $K_\alpha:=H'\oplus\langle v_\alpha\rangle$ $(\alpha\in\mathbb{F}_q^*)$ are distinct elements in Θ with $K_\alpha\cap V=H'$. In fact, since these hyperplanes of $\mathbb{V}$ do not contain the vector r, they are non-degenerate. Moreover since W is negative-type and $H'=W\perp\langle w'\rangle$, we conclude that $K_\alpha=W\perp(W^\perp\cap K_\alpha)$ is negative-type for all $\alpha \in \mathbb{F}_q^*$, and also it follows that they are distinct elements in Θ since we have $v_\alpha + v_\beta = (\alpha+\beta)r$ for $\alpha, \beta \in \mathbb{F}_q^*$. The number of hyperplanes K of $\mathbb{V}$ which include H' is given by $\frac{q^{2m+1}-q^{2m-1}}{q^{2m}-q^{2m-1}} = q+1$. In these $q+1$ hyperplanes of $\mathbb{V}$, $H' \perp \langle r \rangle$ is the only degenerate hyperplane, that is, there are $q-1$ elements K in Θ such that $K \cap V = H'$ and hence each K is written as $K = K_\alpha$ for some $\alpha \in \mathbb{F}_q^*$. By the same reason, there exists an element α_0 in $\mathbb{F}_q^*$ such that

$$U = H \oplus \langle v_{\alpha_0} \rangle.$$

Fix an element α in $\mathbb{F}_q^*$, then since $\mathbb{V} = W \oplus \langle w, w' \rangle \oplus \langle r \rangle$, any vector x in $U \cap K_\alpha$ is uniquely written as

$$x = \xi w + \eta w' + y + \delta r,$$

for some $\xi, \eta, \delta \in \mathbb{F}_q$ and $y \in W$. Then it follows from $U = H \oplus \langle v_{\alpha_0} \rangle$ and $K_\alpha = H' \oplus \langle v_\alpha \rangle$ that

$$\delta = \xi \alpha = \eta \alpha_0. \tag{71}$$

Let w'' be a vector in $U \cap K_\alpha$ such that

$$U \cap K_\alpha = \langle w'' \rangle^\perp \cap U = \langle w'' \rangle^\perp \cap K_\alpha.$$

Since $W^\perp = \langle w, w', r \rangle$, it follows from (71) that

$$w'' = \alpha_0 \epsilon w + \alpha \epsilon w' + \alpha_0 \alpha \epsilon r, \tag{72}$$

for an element $\epsilon \in \mathbb{F}_q$. Since $w'' \neq 0$, we have $\epsilon \neq 0$.

Now suppose $(U, K_\alpha) \in S_i$, then $Q(w'')$ must not be 0, so that

$$\alpha_0^2 \alpha^2 + \alpha_0 \alpha \neq 0,$$

which is equivalent to

$$\alpha \neq \alpha_0^{-1}.$$

We may assume $Q(w'') = 1$ so that

$$\epsilon^2 = \frac{1}{\alpha_0^2 \alpha^2 + \alpha_0 \alpha}.$$

Then we have

$$\frac{1}{\alpha \epsilon} w + \frac{1}{\alpha_0 \epsilon} w' = \frac{1}{\alpha_0 \alpha \epsilon^2} w'' + \frac{1}{\epsilon} r = (\alpha_0 \alpha + 1) w'' + \frac{1}{\epsilon} r. \tag{73}$$

From $f(w, w'') = \alpha \epsilon$, $f(w', w'') = \alpha_0 \epsilon$ and $Q(w) = Q(w') = 0$ it follows that

$$\frac{\alpha_0 \alpha + 1}{\alpha_0 \alpha} = \nu^{i-1} \quad \text{or} \quad \frac{\alpha_0 \alpha + 1}{\alpha_0 \alpha} = \nu^{-(i-1)},$$

or equivalently

$$\alpha_0 \alpha = \lambda_i \quad \text{or} \quad \alpha_0 \alpha = \lambda_i + 1. \tag{74}$$

Conversely if (74) is satisfied, then from (11) we deduce that $(U, K) \in S_i$. Therefore for each 1-dimensional singular subspace $\langle w' \rangle$ in V such that $f(w, w') \neq 0$, there are exactly 2 elements K in Θ such that $(U, K) \in S_i$, $(V, K) \in S_1$ and $V \cap K = H' := \langle w' \rangle^\perp \cap V$. The number of vectors v in V such that $f(v, w) \neq 0$ is $q^{2m} - q^{2m-1}$, and hence the number of 2-dimensional positive-type subspaces of V which include $\langle w \rangle$ is by Proposition 1.1.2 equal to

$$\frac{q^{2m} - q^{2m-1}}{q^2 - q} = q^{2m-2}, \tag{75}$$

which is also equal to the number of 1-dimensional singular subspace $\langle w' \rangle$ in V such that $f(w, w') \neq 0$ since any 2-dimensional positive-type subspace of $\mathbb{V}$ contains two 1-dimensional singular subspaces. Thus the number m_1 of elements K in Θ such that $(U, K) \in S_i$, $(V, K) \in S_1$ and $W = U \cap V \cap K = H \cap K$ is a non-degenerate hyperplane of H, is given by

$$m_1 = 2q^{2m-2}. \tag{76}$$

(b) Suppose $f(w, w') = 0$, then $\langle w, w' \rangle = W^\perp \cap V$ is a singular subspace of W. Hence if an element K in Θ satisfies $U \cap V \cap K = W$, then $U \cap K$ cannot be non-degenerate, since $(U \cap K)^\perp \cap K \subset W^\perp \cap K = \langle w, w' \rangle$. This implies that there is no element K in Θ such that $(U, K) \in S_i$, $(V, K) \in S_1$ and $W = U \cap V \cap K = H \cap K$ is a degenerate hyperplane of H.

Thus by (76)

$$s_{i1}^1 = m_1 = 2q^{2m-2}. \tag{77}$$

(iv) Finally suppose $i = j = k = 1$. Let U and V be elements in Θ such that $(U, V) \in S_1$. We use the same notation as in (iii). In the same way as (iii), fix a vector w' in $V \backslash \langle w \rangle$ with $Q(w') = 1$, and define two subspaces of V as

$$\begin{aligned} H' :=& \langle w' \rangle^\perp \cap V, \\ W :=& H \cap H' = \langle w, w' \rangle^\perp \cap V. \end{aligned}$$

(a) Suppose $f(w, w') \neq 0$, that is, W is a non-degenerate hyperplane of H. We may assume $f(w, w') = 1$ without loss of generality, since $Q(w) = Q(w') = 0$. Define

$$v_\alpha := w + w' + \alpha r,$$

for $\alpha \in \mathbb{F}_q^*$, then as mentioned before, $U = H \oplus \langle v_{\alpha_0} \rangle$ for some $\alpha_0 \in \mathbb{F}_q^*$. Also $K_\alpha := H' \oplus \langle v_\alpha \rangle$ ($\alpha \in \mathbb{F}_q^*$) are distinct elements in Θ with $K_\alpha \cap V = H'$, and each element K in Θ such that $K \cap V = H'$ is written as $K = K_\alpha$ for some $\alpha \in \mathbb{F}_q^*$.

Fix an element α in $\mathbb{F}_q^*$ and let w'' be a vector in $U \cap K_\alpha$ such that $U \cap K_\alpha = \langle w'' \rangle^\perp \cap U = \langle w'' \rangle^\perp \cap K_\alpha$. Then $(U, K) \in S_1$ if and only if $Q(w'') = 0$, which is by (72) equivalent to

$$\alpha_0^2 \alpha^2 + \alpha_0 \alpha = 0,$$

that is, $\alpha = \alpha_0^{-1}$. Therefore for each 1-dimensional singular subspace $\langle w' \rangle$ in V such that $f(w, w') \neq 0$, there is exactly one element K in Θ such that $(U, K) \in S_1$, $(V, K) \in S_1$ and $V \cap K = H' := \langle w' \rangle^\perp \cap V$. Hence it follows from (75) that the number m_1' of elements K in Θ such that $(U, K) \in S_1$, $(V, K) \in S_1$ and $W = U \cap V \cap K = H \cap K$ is a non-degenerate hyperplane of H, is given by

$$m_1' = q^{2m-2}. \tag{78}$$

(b) Suppose $f(w, w') = 0$, that is, W is a degenerate hyperplane of H. In this case any element K in Θ such that $U \cap V \cap K = W$ satisfies $(U, K) \in S_1$, since $(U \cap K)^\perp \cap K \subset W^\perp \cap K = \langle w, w' \rangle$ and $\langle w, w' \rangle$ is a singular subspace.

The number of singular vectors in H is given by

$$q^{m-1} + q^m(q^{m-2} - 1). \tag{79}$$

To show this, let W' be a non-degenerate hyperplane of H so that we have $H = W' \perp \langle w \rangle$. Since W' is negative-type and $Q(w) = 0$, it follows from Lemma 1.1.3(iii) that the number of singular vectors in H is equal to

$$q\{q^{m-2} + q^{m-1}(q^{m-2} - 1)\} = q^{m-1} + q^m(q^{m-2} - 1),$$

as desired. Thus by (79) the number m_2' of elements K in Θ such that $(U, K) \in S_1$, $(V, K) \in S_1$ and $W = U \cap V \cap K = H \cap K$ is a degenerate hyperplane of H, is given by

$$m_2' = \frac{q^{m-1} + q^m(q^{m-2} - 1) - q}{q - 1}(q - 1) = q^{m-1} + q^m(q^{m-2} - 1) - q. \tag{80}$$

(c) We have to count the number m_3' of the elements K in Θ such that $(U, K) \in S_1$, $(V, K) \in S_1$ and $U \cap K = H$. Since there are exactly $\frac{q^{2m+1} - q^{2m-1}}{q^{2m} - q^{2m-1}} - 1 = q$ elements in Θ which include H, m_3' is given by

$$m_3' = q - 2. \tag{81}$$

From (78), (80) and (81) we obtain

$$s_{11}^1 = m_1' + m_2' + m_3' = q^{m-1}(2q^{m-1} - q + 1) - 2. \tag{82}$$

The rest of parameters are directly computed from Proposition 2.1.2:

From (77)

$$s_{11}^k = \frac{h_1}{h_k} s_{k1}^1 = 2q^{m-1}(q^{m-1} - 1) \quad \text{for } 2 \le k \le \tfrac{q}{2}. \tag{83}$$

Also from (70)

$$s_{ij}^1 = \frac{h_j}{h_1} s_{1i}^j = \begin{cases} q^{m-1}(2q^{m-1} + 1) & \text{if } 2 \le i = j \le \frac{q}{2}, \\ 2q^{2m-2} & \text{if } 2 \le i, j \le \frac{q}{2} \text{ and } i \ne j. \end{cases} \tag{84}$$

To summarize:

Lemma 2.2.1. *The intersection numbers s_{ij}^k of $\mathfrak{X}(GO_{2m+1}(q), \Theta)$ are given as follows.*

$$s_{ij}^1 = s_{ji}^1 = \begin{cases} q^{m-1}(2q^{m-1} - q + 1) - 2 & \textit{if } \ i = j = 1, \\ q^{m-1}(2q^{m-1} + 1) & \textit{if } \ 2 \le i = j \le \frac{q}{2}, \\ 2q^{2m-2} & \textit{if } \ 1 \le i < j \le \frac{q}{2}, \end{cases}$$

$$s_{1j}^k = s_{j1}^k = \begin{cases} (2q^{m-1} + 1)(q^{m-1} - 1) & \textit{if } \ 2 \le j = k \le \frac{q}{2}, \\ 2q^{m-1}(q^{m-1} - 1) & \textit{if } \ 1 \le j \le \frac{q}{2}, \ 2 \le k \le \frac{q}{2}, \ j \ne k. \end{cases}$$

For other $2 \le i, j, k \le \frac{q}{2}$,

$$s_{ij}^k = \begin{cases} q^{m-1}(2q^{m-1} - 1) & \textit{if } \mu_i^2 + \mu_j^2 + \mu_k^2 = 0 \ \textit{ and } \ \phi\left(\frac{1}{\mu_j \mu_k}\right) = 1, \\ q^{m-1}(2q^{m-1} + 1) & \textit{if } \mu_i^2 + \mu_j^2 + \mu_k^2 = 0 \ \textit{ and } \ \phi\left(\frac{1}{\mu_j \mu_k}\right) = 0, \\ 2q^{m-1}(q^{m-1} - 1) & \textit{if } \mu_i^2 + \mu_j^2 + \mu_k^2 \ne 0 \ \textit{ and } \ \phi(\kappa_{ijk}) = \phi(\kappa_{ijk}') = 1, \\ 2q^{m-1}(q^{m-1} + 1) & \textit{if } \mu_i^2 + \mu_j^2 + \mu_k^2 \ne 0 \ \textit{ and } \ \phi(\kappa_{ijk}) = \phi(\kappa_{ijk}') = 0, \\ 2q^{2m-2} & \textit{if } \mu_i^2 + \mu_j^2 + \mu_k^2 \ne 0 \ \textit{ and } \ \{\phi(\kappa_{ijk}), \phi(\kappa_{ijk}')\} = \{0, 1\}. \end{cases}$$

3 Character Tables

3.1 The Character Tables of $\mathfrak{X}(GO_{2m+1}(q), \Omega_{2m+1}(q))$

In this subsection, we determine the character table of $\mathfrak{X}(GO_{2m+1}(q), \Omega_{2m+1}(q))$ explicitly. Our account follows Bannai-Hao-Song [2, §6.] in all essential points. Namely, we prove that the character table of $\mathfrak{X}(GO_{2m+1}(q), \Omega_{2m+1}(q))$ is controlled by that of $\mathfrak{X}(GO_3(q), \Omega_3(q))$.

First of all, we prove the following lemma which shows the relation between the sets of parameters of $\mathfrak{X}(GO_{2m+1}(q), \Omega_{2m+1}(q))$ and $\mathfrak{X}(GO_3(q), \Omega_3(q))$:

Lemma 3.1.1. *Let $\{a_{ij}^k\}$ denotes the set of the intersection numbers of $\mathfrak{X}(GO_3(q), \Omega_3(q))$. Then*

$$p_{11}^1 = 2q^{m-1}(q^{m-1} - 1) + q^{m-1}(a_{11}^1 + 2) - 2$$
$$p_{1j}^j = p_{j1}^j = 2q^{m-1}(q^{m-1} - 1) + q^{m-1}(a_{1j}^j + 1) - 1 \quad \textit{for } \ 2 \le j \le \tfrac{q}{2},$$

for other $1 \le i, j, k \le \frac{q}{2}$,

$$p_{ij}^k = 2q^{m-1}(q^{m-1} - 1) + q^{m-1} a_{ij}^k.$$

Proof. For $i = j = k = 1$, from $a_{11}^1 = q - 1$ we have

$$\begin{aligned} p_{11}^1 &= q^{m-1}(2q^{m-1} + q - 1) - 2 = 2q^{m-1}(q^{m-1} - 1) + q^m + q^{m-1} - 2 \\ &= 2q^{m-1}(q^{m-1} - 1) + q^{m-1}(a_{11}^1 + 2) - 2. \end{aligned}$$

For $2 \le i = j \le \frac{q}{2}$ and $k = 1$, from $a_{ii}^1 = 1$ we have

$$\begin{aligned} p_{ii}^1 &= q^{m-1}(2q^{m-1} - 1) = 2q^{m-1}(q^{m-1} - 1) + q^{m-1} \\ &= 2q^{m-1}(q^{m-1} - 1) + q^{m-1}a_{ii}^1. \end{aligned}$$

For $1 \le i < j \le \frac{q}{2}$ and $k = 1$, from $a_{ij}^1 = a_{ji}^1 = 2$ we have

$$\begin{aligned} p_{ij}^1 = p_{ji}^1 &= 2q^{2m-2} = 2q^{m-1}(q^{m-1} - 1) + 2q^{m-1} \\ &= 2q^{m-1}(q^{m-1} - 1) + q^{m-1}a_{ij}^1. \end{aligned}$$

For $i = 1$ and $2 \le j = k \le \frac{q}{2}$, from $a_{1j}^j = a_{j1}^j = 2$ we have

$$\begin{aligned} p_{1j}^j = p_{j1}^j &= (2q^{m-1} - 1)(q^{m-1} + 1) = 2q^{m-1}(q^{m-1} - 1) + 3q^{m-1} - 1 \\ &= 2q^{m-1}(q^{m-1} - 1) + q^{m-1}(a_{1j}^j + 1) - 1. \end{aligned}$$

For $i = 1$, $1 \le j \le \frac{q}{2}$, $2 \le k \le \frac{q}{2}$ and $j \ne k$, from $a_{1j}^k = a_{j1}^k = 4$ we have

$$\begin{aligned} p_{1j}^k = p_{j1}^k &= 2q^{m-1}(q^{m-1} + 1) = 2q^{m-1}(q^{m-1} - 1) + 4q^{m-1} \\ &= 2q^{m-1}(q^{m-1} - 1) + q^{m-1}a_{1j}^k. \end{aligned}$$

For $2 \le i, j, k \le \frac{q}{2}$, $\mu_i^2 + \mu_j^2 + \mu_k^2 = 0$ and $\phi\left(\frac{1}{\mu_j \mu_k}\right) = 1$, from $a_{ij}^k = 3$ we have

$$\begin{aligned} p_{ij}^k &= q^{m-1}(2q^{m-1} + 1) = 2q^{m-1}(q^{m-1} - 1) + 3q^{m-1} \\ &= 2q^{m-1}(q^{m-1} - 1) + q^{m-1}a_{ij}^k. \end{aligned}$$

For $2 \le i, j, k \le \frac{q}{2}$, $\mu_i^2 + \mu_j^2 + \mu_k^2 = 0$ and $\phi\left(\frac{1}{\mu_j \mu_k}\right) = 0$, from $a_{ij}^k = 1$ we have

$$\begin{aligned} p_{ij}^k &= q^{m-1}(2q^{m-1} - 1) = 2q^{m-1}(q^{m-1} - 1) + q^{m-1} \\ &= 2q^{m-1}(q^{m-1} - 1) + q^{m-1}a_{ij}^k. \end{aligned}$$

For $2 \le i, j, k \le \frac{q}{2}$, $\mu_i^2 + \mu_j^2 + \mu_k^2 \ne 0$ and $\phi(\kappa_{ijk}) = \phi(\kappa'_{ijk}) = 1$, from $a_{ij}^k = 4$ we have

$$\begin{aligned} p_{ij}^k &= 2q^{m-1}(q^{m-1} + 1) = 2q^{m-1}(q^{m-1} - 1) + 4q^{m-1} \\ &= 2q^{m-1}(q^{m-1} - 1) + q^{m-1}a_{ij}^k. \end{aligned}$$

For $2 \le i, j, k \le \frac{q}{2}$, $\mu_i^2 + \mu_j^2 + \mu_k^2 \ne 0$ and $\phi(\kappa_{ijk}) = \phi(\kappa'_{ijk}) = 0$, from $a_{ij}^k = 0$ we have

$$\begin{aligned} p_{ij}^k &= 2q^{m-1}(q^{m-1} - 1) \\ &= 2q^{m-1}(q^{m-1} - 1) + q^{m-1}a_{ij}^k. \end{aligned}$$

For $2 \le i, j, k \le \frac{q}{2}$, $\mu_i^2 + \mu_j^2 + \mu_k^2 \ne 0$ and $\{\phi(\kappa_{ijk}), \phi(\kappa'_{ijk})\} = \{0, 1\}$, from $a_{ij}^k = 2$ we have

$$\begin{aligned} p_{ij}^k &= 2q^{2m-2} = 2q^{m-1}(q^{m-1} - 1) + 2q^{m-1} \\ &= 2q^{m-1}(q^{m-1} - 1) + q^{m-1}a_{ij}^k. \end{aligned}$$

This proves Lemma 3.1.1. □

It is known that the character table $\tilde{P}^+ = (\tilde{p}_j(i))$ of $\mathfrak{X}(GO_3(q), \Omega_3(q))$ is given as follows (cf. Tanaka [14]):

$$\tilde{P}^+ = \begin{bmatrix} 1 & 2(q-1) & (q-1) & \dots & (q-1) \\ 1 & q-3 & -2 & \dots & -2 \\ 1 & -2 & & & \\ \vdots & \vdots & & (\chi_{ij})_{2\le i,j\le \frac{q}{2}} & \\ 1 & -2 & & & \end{bmatrix}, \tag{85}$$

for suitable $\chi_{ij} \in \mathbb{Q}(\theta)$ with $\theta = \exp(\frac{2\pi i}{q-1})$. The values of the entries χ_{ij} are slightly complicated. The explicit description of these values are given in [14].

Theorem 3.1.2. *The character table $P^+ = (p_j(i))$ of $\mathfrak{X}(GO_{2m+1}(q), \Omega_{2m+1}(q))$ is described as*

$$P^+ = \begin{bmatrix} 1 & (q^{m-1}+1)(q^m-1) & q^{m-1}(q^m-1) & \dots & q^{m-1}(q^m-1) \\ 1 & (q-2)q^{m-1}-1 & -2q^{m-1} & \dots & -2q^{m-1} \\ 1 & -(q^{m-1}+1) & & & \\ \vdots & \vdots & & (q^{m-1}\chi_{ij})_{2\le i,j\le \frac{q}{2}} & \\ 1 & -(q^{m-1}+1) & & & \end{bmatrix}.$$

That is,

$$\begin{aligned} &p_0(i) = 1 \quad \text{for } 0 \le i \le \tfrac{q}{2} \\ &p_j(0) = k_j \quad \text{for } 0 \le j \le \tfrac{q}{2} \\ &p_1(i) = q^{m-1}\tilde{p}_1(i) + q^{m-1} - 1 \quad \text{for } 1 \le i \le \tfrac{q}{2} \\ &p_j(i) = q^{m-1}\tilde{p}_j(i) \quad \text{for } 1 \le i \le \tfrac{q}{2},\ 2 \le j \le \tfrac{q}{2} \end{aligned}$$

Proof. The transposition of each row of the character table of $\mathfrak{X}(GO_{2m+1}(q), \Omega_{2m+1}(q))$ is a common right eigenvector of the *intersection matrices* $B_0, B_1, \dots, B_{\frac{q}{2}}$, where B_i is the matrix whose (j,k)-entry is p^k_{ij} (cf. Bannai-Ito [4, p.91, Proposition 5.3.]). Thus, we have only to show that the following equality:

$$B_i \begin{pmatrix} p_0(l) \\ p_1(l) \\ \vdots \\ p_{\frac{q}{2}}(l) \end{pmatrix} = p_i(l) \begin{pmatrix} p_0(l) \\ p_1(l) \\ \vdots \\ p_{\frac{q}{2}}(l) \end{pmatrix} \tag{86}$$

for all i and l.

(i) Suppose first $i = j = 1$ and $1 \le l \le \frac{q}{2}$, then using the equality (1) and Lemma 3.1.1 we see that

$$\begin{aligned} \sum_{\alpha=0}^{\frac{q}{2}} p^\alpha_{11} p_\alpha(l) &= p^0_{11} p_0(l) + p^1_{11} p_1(l) + \sum_{\alpha=2}^{\frac{q}{2}} p^\alpha_{11} p_\alpha(l) \\ &= (q^{m-1}+1)(q^m-1) \\ &\quad + 2q^{2m-2}(q^{m-1}-1)\tilde{p}_1(l) + q^{2m-2}a^1_{11}\tilde{p}_1(l) + 2q^{m-1}(q^{m-1}-1)\tilde{p}_1(l) \\ &\quad + \{q^{m-1}(2q^{m-1}+q-1)-2\}(q^{m-1}-1) \\ &\quad + 2q^{2m-2}(q^{m-1}-1)\sum_{\alpha=0}^{\frac{q}{2}} \tilde{p}_\alpha(l) + q^{2m-2}\sum_{\alpha=0}^{\frac{q}{2}} a^\alpha_{11}\tilde{p}_\alpha(l) \\ &\quad - 2q^{2m-2}(q^{m-1}-1) - 2q^{2m-2}(q^{m-1}-1)\tilde{p}_1(l) \\ &\quad - 2q^{2m-2}(q-1) - q^{2m-2}a^1_{11}\tilde{p}_1(l) \qquad \text{by } a^0_{11} = 2(q-1), \\ &= q^{2m-2}\tilde{p}_1(l)^2 + 2q^{m-1}(q^{m-1}-1)\tilde{p}_1(l) + (q^{m-1}-1)^2 \\ &= \{q^{m-1}\tilde{p}_1(l) + q^{m-1} - 1\}^2 \end{aligned}$$

$$= p_1(l)^2.$$

(ii) Suppose $i = 1$, $1 < j \le \frac{q}{2}$ and $1 \le l \le \frac{q}{2}$, then

$$\begin{aligned}
\sum_{\alpha=0}^{\frac{q}{2}} p_{1j}^{\alpha} p_{\alpha}(l) &= p_{1j}^{0} p_0(l) + p_{1j}^{1} p_1(l) + p_{1j}^{j} p_j(l) + \sum_{\substack{\alpha=2 \\ \alpha \neq j}}^{\frac{q}{2}} p_{1j}^{\alpha} p_{\alpha}(l) \\
&= 2q^{2m-2}(q^{m-1}-1)\tilde{p}_1(l) + q^{2m-2} a_{1j}^{1} \tilde{p}_1(l) + 2q^{2m-2}(q^{m-1}-1) \\
&\quad + 2q^{2m-2}(q^{m-1}-1)\tilde{p}_j(l) + q^{2m-2} a_{1j}^{j} \tilde{p}_j(l) + q^{m-1}(q^{m-1}-1)\tilde{p}_j(l) \\
&\quad + 2q^{2m-2}(q^{m-1}-1) \sum_{\alpha=0}^{\frac{q}{2}} \tilde{p}_{\alpha}(l) + q^{2m-2} \sum_{\alpha=0}^{\frac{q}{2}} a_{1j}^{\alpha} \tilde{p}_{\alpha}(l) \\
&\quad - 2q^{2m-2}(q^{m-1}-1) - 2q^{2m-2}(q^{m-1}-1)\tilde{p}_1(l) - 2q^{2m-2}(q^{m-1}-1)\tilde{p}_j(l) \\
&\quad - q^{2m-2} a_{1j}^{0} \tilde{p}_0(l) - q^{2m-2} a_{1j}^{1} \tilde{p}_1(l) - q^{2m-2} a_{1j}^{j} \tilde{p}_j(l) \\
&= q^{2m-2} \tilde{p}_1(l) \tilde{p}_j(l) + q^{m-1}(q^{m-1}-1)\tilde{p}_j(l) \\
&= \{q^{m-1}\tilde{p}_1(l) + q^{m-1} - 1\} q^{m-1} \tilde{p}_j(l) \\
&= p_1(l) p_j(l).
\end{aligned}$$

(iii) Suppose $1 < i \le \frac{q}{2}$, $j = 1$ and $1 \le l \le \frac{q}{2}$, then

$$\begin{aligned}
\sum_{\alpha=0}^{\frac{q}{2}} p_{i1}^{\alpha} p_{\alpha}(l) &= p_{i1}^{0} p_0(l) + p_{i1}^{1} p_1(l) + p_{i1}^{i} p_i(l) + \sum_{\substack{\alpha=2 \\ \alpha \neq i}}^{\frac{q}{2}} p_{i1}^{\alpha} p_{\alpha}(l) \\
&= 2q^{2m-2}(q^{m-1}-1)\tilde{p}_1(l) + q^{2m-2} a_{i1}^{1} \tilde{p}_1(l) + 2q^{2m-2}(q^{m-1}-1) \\
&\quad + 2q^{2m-2}(q^{m-1}-1)\tilde{p}_i(l) + q^{2m-2} a_{i1}^{i} \tilde{p}_i(l) + q^{m-1}(q^{m-1}-1)\tilde{p}_i(l) \\
&\quad + 2q^{2m-2}(q^{m-1}-1) \sum_{\alpha=0}^{\frac{q}{2}} \tilde{p}_{\alpha}(l) + q^{2m-2} \sum_{\alpha=0}^{\frac{q}{2}} a_{i1}^{\alpha} \tilde{p}_{\alpha}(l) \\
&\quad - 2q^{2m-2}(q^{m-1}-1) - 2q^{2m-2}(q^{m-1}-1)\tilde{p}_1(l) - 2q^{2m-2}(q^{m-1}-1)\tilde{p}_i(l) \\
&\quad - q^{2m-2} a_{i1}^{0} \tilde{p}_0(l) - q^{2m-2} a_{i1}^{1} \tilde{p}_1(l) - q^{2m-2} a_{i1}^{i} \tilde{p}_i(l) \\
&= q^{2m-2} \tilde{p}_i(l) \tilde{p}_1(l) + q^{m-1}(q^{m-1}-1)\tilde{p}_i(l) \\
&= q^{m-1} \tilde{p}_i(l) \{q^{m-1}\tilde{p}_1(l) + q^{m-1} - 1\} \\
&= p_i(l) p_1(l).
\end{aligned}$$

(iv) Suppose $1 < i = j \le \frac{q}{2}$ and $1 \le l \le \frac{q}{2}$, then

$$\begin{aligned}
\sum_{\alpha=0}^{\frac{q}{2}} p_{ii}^{\alpha} p_{\alpha}(l) &= p_{ii}^{0} p_0(l) + p_{ii}^{1} p_1(l) + \sum_{\alpha=2}^{\frac{q}{2}} p_{ii}^{\alpha} p_{\alpha}(l) \\
&= q^{m-1}(q^m - 1) \\
&\quad + 2q^{2m-2}(q^{m-1}-1)\tilde{p}_1(l) + q^{2m-2} a_{ii}^{1} \tilde{p}_1(l) + q^{m-1}(2q^{m-1}-1)(q^{m-1}-1) \\
&\quad + 2q^{2m-2}(q^{m-1}-1) \sum_{\alpha=0}^{\frac{q}{2}} \tilde{p}_{\alpha}(l) + q^{2m-2} \sum_{\alpha=0}^{\frac{q}{2}} a_{ii}^{\alpha} \tilde{p}_{\alpha}(l) \\
&\quad - 2q^{2m-2}(q^{m-1}-1) - 2q^{2m-2}(q^{m-1}-1)\tilde{p}_1(l) \\
&\quad - q^{2m-2}(q-1) - q^{2m-2} a_{ii}^{1} \tilde{p}_1(l) \qquad \text{by } a_{ii}^{0} = q-1, \\
&= q^{2m-2} \tilde{p}_i(l)^2 \\
&= p_i(l)^2.
\end{aligned}$$

(v)Finally, suppose $1 < i, j \leq \frac{q}{2}$, $i \neq j$ and $1 \leq l \leq \frac{q}{2}$, then

$$\begin{aligned}
\sum_{\alpha=0}^{\frac{q}{2}} p_{ij}^{\alpha} p_{\alpha}(l) &= p_{ij}^{0} p_0(l) + p_{ij}^{1} p_1(l) + \sum_{\alpha=2}^{\frac{q}{2}} p_{ij}^{\alpha} p_{\alpha}(l) \\
&= 2q^{2m-2}(q^{m-1}-1)\tilde{p}_1(l) + q^{2m-2} a_{ij}^{1} \tilde{p}_1(l) + 2q^{2m-2}(q^{m-1}-1) \\
&\quad + 2q^{2m-2}(q^{m-1}-1) \sum_{\alpha=0}^{\frac{q}{2}} \tilde{p}_{\alpha}(l) + q^{2m-2} \sum_{\alpha=0}^{\frac{q}{2}} a_{ij}^{\alpha} \tilde{p}_{\alpha}(l) \\
&\quad - 2q^{2m-2}(q^{m-1}-1) - 2q^{2m-2}(q^{m-1}-1)\tilde{p}_1(l) \\
&\quad - q^{2m-2} a_{ij}^{0} - q^{2m-2} a_{ij}^{1} \tilde{p}_1(l) \\
&= q^{2m-2} \tilde{p}_i(l) \tilde{p}_j(l) \\
&= p_i(l) p_j(l).
\end{aligned}$$

This completes the proof of Theorem 3.1.2. □

3.2 The Character Tables of $\mathfrak{X}(GO_{2m+1}(q), \Theta_{2m+1}(q))$

We have shown that the character table of $\mathfrak{X}(GO_{2m+1}(q), \Omega_{2m+1}(q))$ is essentially controlled by that of a smaller association scheme $\mathfrak{X}(GO_3(q), \Omega_3(q))$, by the replacement $q \to q^{m-1}$. Although it is possible to calculate the character table of $\mathfrak{X}(GO_{2m+1}(q), \Theta_{2m+1}(q))$ in the same way, we observe a similar kind of phenomenon which is called an *Ennola type duality* (cf. Bannai-Kwok-Song [6]), that is, we will show that the character table of $\mathfrak{X}(GO_{2m+1}(q), \Theta_{2m+1}(q))$ is essentially obtained by that of $\mathfrak{X}(GO_{2m+1}(q), \Omega_{2m+1}(q))$, by the replacement $q \to -q$. Consequently it follows that the charactr table of $\mathfrak{X}(GO_{2m+1}(q), \Theta_{2m+1}(q))$ is controlled by that of $\mathfrak{X}(GO_3(q), \Theta_3(q))$.

The following lemma shows the relation between the parameters of $\mathfrak{X}(GO_{2m+1}(q), \Theta_{2m+1}(q))$ and those of $\mathfrak{X}(GO_{2m+1}(q), \Omega_{2m+1}(q))$ for $m \geq 2$, also the relation between the parameters of $\mathfrak{X}(GO_3(q), \Theta_3(q))$ and those of $\mathfrak{X}(GO_3(q), \Omega_3(q))$. (Notice that $\mathfrak{X}(GO_3(q), \Theta_3(q))$ is of class $\frac{q}{2} - 1$ while $\mathfrak{X}(GO_{2m+1}(q), \Theta_{2m+1}(q))$ is of class $\frac{q}{2}$ for $m \geq 2$.)

Lemma 3.2.1. *Let $\{b_{ij}^k\}$ denotes the set of the intersection numbers of $\mathfrak{X}(GO_3(q), \Theta_3(q))$. Then for $m \geq 2$*

$$\begin{aligned}
s_{11}^{1} &= 4q^{2m-2} - p_{11}^{1} - 4 \\
s_{1j}^{j} &= s_{j1}^{j} = 4q^{2m-2} - p_{1j}^{j} - 2 \quad \textit{for} \ \ 2 \leq j \leq \tfrac{q}{2},
\end{aligned}$$

for other $1 \leq i, j, k \leq \frac{q}{2}$,

$$s_{ij}^{k} = 4q^{2m-2} - p_{ij}^{k}.$$

Also

$$b_{ij}^{k} = 4 - a_{ij}^{k},$$

for $2 \leq i, j, k \leq \frac{q}{2}$.

Proof. For $i = j = k = 1$, from $p_{11}^1 = q^{m-1}(2q^{m-1} + q - 1) - 2$ we have

$$\begin{aligned}
s_{11}^{1} &= q^{m-1}(2q^{m-1} - q + 1) - 2 = 4q^{2m-2} - q^{m-1}(2q^{m-1} + q - 1) - 2 \\
&= 4q^{2m-2} - p_{11}^{1} - 4.
\end{aligned}$$

For $2 \leq i = j \leq \frac{q}{2}$ and $k = 1$, from $p_{ii}^1 = q^{m-1}(2q^{m-1} - 1)$ we have

$$\begin{aligned}
s_{ii}^{1} &= q^{m-1}(2q^{m-1} + 1) = 4q^{2m-2} - q^{m-1}(2q^{m-1} - 1) \\
&= 4q^{2m-2} - p_{ii}^{1}.
\end{aligned}$$

For $1 \leq i < j \leq \frac{q}{2}$ and $k = 1$, from $p_{ij}^1 = p_{ji}^1 = 2q^{2m-2}$ we have

$$\begin{aligned}
s_{ij}^{1} &= s_{ji}^{1} = 2q^{2m-2} = 4q^{2m-2} - 2q^{2m-2} \\
&= 4q^{2m-2} - p_{ij}^{1}.
\end{aligned}$$

For $i = 1$ and $2 \leq j = k \leq \frac{q}{2}$, from $p_{1j}^j = p_{j1}^j = (2q^{m-1} - 1)(q^{m-1} + 1)$ we have

$$\begin{aligned} s_{1j}^j = s_{j1}^j &= (2q^{m-1} + 1)(q^{m-1} - 1) = 4q^{2m-2} - (2q^{2m-2} + q^{m-1} - 1) - 2 \\ &= 4q^{2m-2} - p_{1j}^j - 2. \end{aligned}$$

For $i = 1$, $1 \leq j \leq \frac{q}{2}$, $2 \leq k \leq \frac{q}{2}$ and $j \neq k$, from $p_{1j}^k = p_{j1}^k = 2q^{m-1}(q^{m-1} + 1)$ we have

$$\begin{aligned} s_{1j}^k = s_{j1}^k &= 2q^{m-1}(q^{m-1} - 1) = 4q^{2m-2} - 2q^{m-1}(q^{m-1} + 1) \\ &= 4q^{2m-2} - p_{1j}^k. \end{aligned}$$

For $2 \leq i, j, k \leq \frac{q}{2}$, $\mu_i^2 + \mu_j^2 + \mu_k^2 = 0$ and $\phi\left(\frac{1}{\mu_j \mu_k}\right) = 1$, from $p_{ij}^k = q^{m-1}(2q^{m-1} + 1)$ we have

$$\begin{aligned} s_{ij}^k &= q^{m-1}(2q^{m-1} - 1) = 4q^{2m-2} - q^{m-1}(2q^{m-1} + 1) \\ &= 4q^{2m-2} - p_{ij}^k. \end{aligned}$$

For $2 \leq i, j, k \leq \frac{q}{2}$, $\mu_i^2 + \mu_j^2 + \mu_k^2 = 0$ and $\phi\left(\frac{1}{\mu_j \mu_k}\right) = 0$, from $p_{ij}^k = q^{m-1}(2q^{m-1} - 1)$ we have

$$\begin{aligned} s_{ij}^k &= q^{m-1}(2q^{m-1} + 1) = 4q^{2m-2} - q^{m-1}(2q^{m-1} - 1) \\ &= 4q^{2m-2} - p_{ij}^k. \end{aligned}$$

For $2 \leq i, j, k \leq \frac{q}{2}$, $\mu_i^2 + \mu_j^2 + \mu_k^2 \neq 0$ and $\phi(\kappa_{ijk}) = \phi(\kappa'_{ijk}) = 1$, from $p_{ij}^k = 2q^{m-1}(q^{m-1} + 1)$ we have

$$\begin{aligned} s_{ij}^k &= 2q^{m-1}(q^{m-1} - 1) = 4q^{2m-2} - 2q^{m-1}(q^{m-1} + 1) \\ &= 4q^{2m-2} - p_{ij}^k. \end{aligned}$$

For $2 \leq i, j, k \leq \frac{q}{2}$, $\mu_i^2 + \mu_j^2 + \mu_k^2 \neq 0$ and $\phi(\kappa_{ijk}) = \phi(\kappa'_{ijk}) = 0$, from $p_{ij}^k = 2q^{m-1}(q^{m-1} - 1)$ we have

$$\begin{aligned} s_{ij}^k &= 2q^{m-1}(q^{m-1} + 1) = 4q^{2m-2} - 2q^{m-1}(q^{m-1} - 1) \\ &= 4q^{2m-2} - p_{ij}^k. \end{aligned}$$

For $2 \leq i, j, k \leq \frac{q}{2}$, $\mu_i^2 + \mu_j^2 + \mu_k^2 \neq 0$ and $\{\phi(\kappa_{ijk}), \phi(\kappa'_{ijk})\} = \{0, 1\}$, from $p_{ij}^k = 2q^{2m-2}$ we have

$$\begin{aligned} s_{ij}^k &= 2q^{2m-2} = 4q^{2m-2} - 2q^{2m-2} \\ &= 4q^{2m-2} - p_{ij}^k. \end{aligned}$$

This proves Lemma 3.2.1.

□

Theorem 3.2.2. *For $m \geq 2$, the character table $P^- = (s_j(i))$ of $\mathfrak{X}(GO_{2m+1}(q), \Theta)$ is described as*

$$P^- = \begin{bmatrix} 1 & (q^{m-1} - 1)(q^m + 1) & q^{m-1}(q^m + 1) & \dots & q^{m-1}(q^m + 1) \\ 1 & -(q-2)q^{m-1} - 1 & 2q^{m-1} & \dots & 2q^{m-1} \\ 1 & (q^{m-1} - 1) & & & \\ \vdots & \vdots & & (-q^{m-1}\chi_{ij})_{2 \leq i,j \leq \frac{q}{2}} & \\ 1 & (q^{m-1} - 1) & & & \end{bmatrix}$$

That is,

$$\begin{aligned} & s_0(i) = 1 \quad \textit{for } 0 \leq i \leq \tfrac{q}{2} \\ & s_j(0) = h_j \quad \textit{for } 0 \leq j \leq \tfrac{q}{2} \\ & s_1(i) = -p_1(i) - 2 \quad \textit{for } 1 \leq i \leq \tfrac{q}{2} \\ & s_j(i) = -p_j(i) \quad \textit{for } 1 \leq i \leq \tfrac{q}{2},\ 2 \leq j \leq \tfrac{q}{2} \end{aligned}$$

Proof. In the same way as the proof of Theorem 3.1.2, we verify the following equality:

$$\sum_{\alpha=0}^{\frac{q}{2}} s_{ij}^{\alpha} s_{\alpha}(l) = s_i(l)s_j(l) \tag{87}$$

for all $i, j, l \in \{0, 1, \ldots, \frac{q}{2}\}$.

(i) Suppose first $i = j = 1$ and $1 \le l \le \frac{q}{2}$, then using the equality (1) and Lemma 3.2.1 we see that

$$\begin{aligned}
\sum_{\alpha=0}^{\frac{q}{2}} s_{11}^{\alpha} s_{\alpha}(l) &= s_{11}^{0} s_0(l) + s_{11}^{1} s_1(l) + \sum_{\alpha=2}^{\frac{q}{2}} s_{11}^{\alpha} s_{\alpha}(l) \\
&= (q^{m-1}-1)(q^m+1) \\
&\quad - 4q^{2m-2}p_1(l) + p_{11}^{1}p_1(l) + 4p_1(l) - 2\{q^{m-1}(2q^{m-1}-q+1)-2\} \\
&\quad - 4q^{2m-2}\sum_{\alpha=0}^{\frac{q}{2}} p_{\alpha}(l) + \sum_{\alpha=0}^{\frac{q}{2}} p_{11}^{\alpha} p_{\alpha}(l) \\
&\quad + 4q^{2m-2} + 4q^{2m-2}p_1(l) \\
&\quad - (q^{m-1}+1)(q^m-1) - p_{11}^{1}p_1(l) \qquad \text{by } p_{11}^{0} = (q^{m-1}+1)(q^m-1), \\
&= p_1(l)^2 + 4p_1(l) + 4 \\
&= (p_1(l)+2)^2 \\
&= s_1(l)^2.
\end{aligned}$$

(ii) Suppose $i = 1$, $1 < j \le \frac{q}{2}$ and $1 \le l \le \frac{q}{2}$, then

$$\begin{aligned}
\sum_{\alpha=0}^{\frac{q}{2}} s_{1j}^{\alpha} s_{\alpha}(l) &= s_{1j}^{0} s_0(l) + s_{1j}^{1} s_1(l) + s_{1j}^{j} s_j(l) + \sum_{\substack{\alpha=2\\ \alpha\neq j}}^{\frac{q}{2}} s_{1j}^{\alpha} s_{\alpha}(l) \\
&= -4q^{2m-2}p_1(l) + p_{1j}^{1}p_1(l) - 4q^{2m-2} \\
&\quad - 4q^{2m-2}p_j(l) + p_{1j}^{j}p_j(l) + 2p_j(l) \\
&\quad - 4q^{2m-2}\sum_{\alpha=0}^{\frac{q}{2}} p_{\alpha}(l) + \sum_{\alpha=0}^{\frac{q}{2}} p_{1j}^{\alpha} p_{\alpha}(l) \\
&\quad + 4q^{2m-2} + 4q^{2m-2}p_1(l) + 4q^{2m-2}p_j(l) \\
&\quad - p_{1j}^{0}p_0(l) - p_{1j}^{1}p_1(l) - p_{1j}^{j}p_j(l) \\
&= p_1(l)p_j(l) + 2p_j(l) \\
&= (p_1(l)+2)p_j(l) \\
&= s_1(l)s_j(l).
\end{aligned}$$

(iii) Suppose $1 < i \le \frac{q}{2}$, $j = 1$ and $1 \le l \le \frac{q}{2}$, then

$$\begin{aligned}
\sum_{\alpha=0}^{\frac{q}{2}} s_{i1}^{\alpha} s_{\alpha}(l) &= s_{i1}^{0} s_0(l) + s_{i1}^{1} s_1(l) + s_{i1}^{i} s_i(l) + \sum_{\substack{\alpha=2\\ \alpha\neq i}}^{\frac{q}{2}} s_{i1}^{\alpha} s_{\alpha}(l) \\
&= -4q^{2m-2}p_1(l) + p_{i1}^{1}p_1(l) - 4q^{2m-2} \\
&\quad - 4q^{2m-2}p_i(l) + p_{i1}^{i}p_i(l) + 2p_i(l) \\
&\quad - 4q^{2m-2}\sum_{\alpha=0}^{\frac{q}{2}} p_{\alpha}(l) + \sum_{\alpha=0}^{\frac{q}{2}} p_{i1}^{\alpha} p_{\alpha}(l)
\end{aligned}$$

$$
\begin{aligned}
&+4q^{2m-2}+4q^{2m-2}p_1(l)+4q^{2m-2}p_i(l)\\
&-p_{i1}^0p_0(l)-p_{i1}^1p_1(l)-p_{i1}^ip_i(l)\\
&=p_i(l)p_1(l)+2p_i(l)\\
&=p_i(l)(p_1(l)+2)\\
&=s_i(l)s_1(l).
\end{aligned}
$$

(iv) Suppose $1 < i = j \leq \frac{q}{2}$ and $1 \leq l \leq \frac{q}{2}$, then

$$
\begin{aligned}
\sum_{\alpha=0}^{\frac{q}{2}} s_{ii}^{\alpha}s_{\alpha}(l) &= s_{ii}^0s_0(l)+s_{ii}^1s_1(l)+\sum_{\alpha=2}^{\frac{q}{2}} s_{ii}^{\alpha}s_{\alpha}(l)\\
&= q^{m-1}(q^m+1)\\
&\quad -4q^{2m-2}p_1(l)+p_{ii}^1p_1(l)-2q^{m-1}(2q^{m-1}+1)\\
&\quad -4q^{2m-2}\sum_{\alpha=0}^{\frac{q}{2}}p_{\alpha}(l)+\sum_{\alpha=0}^{\frac{q}{2}}p_{ii}^{\alpha}p_{\alpha}(l)\\
&\quad +4q^{2m-2}+4q^{2m-2}p_1(l)\\
&\quad -q^{m-1}(q^m-1)-p_{ii}^1p_1(l) \qquad \text{by } p_{ii}^0=q^{m-1}(q^m-1),\\
&= p_i(l)^2\\
&= s_i(l)^2.
\end{aligned}
$$

(v)Finally, suppose $1 < i,j \leq \frac{q}{2}$, $i \neq j$ and $1 \leq l \leq \frac{q}{2}$, then

$$
\begin{aligned}
\sum_{\alpha=0}^{\frac{q}{2}} s_{ij}^{\alpha}s_{\alpha}(l) &= s_{ij}^0s_0(l)+s_{ij}^1s_1(l)+\sum_{\alpha=2}^{\frac{q}{2}} s_{ij}^{\alpha}s_{\alpha}(l)\\
&= -4q^{2m-2}p_1(l)+p_{ij}^1p_1(l)-4q^{2m-2}\\
&\quad -4q^{2m-2}\sum_{\alpha=0}^{\frac{q}{2}}p_{\alpha}(l)+\sum_{\alpha=0}^{\frac{q}{2}}p_{ij}^{\alpha}p_{\alpha}(l)\\
&\quad +4q^{2m-2}+4q^{2m-2}p_1(l)\\
&\quad -p_{ij}^0-p_{ij}^1p_1(l)\\
&= p_i(l)p_j(l)\\
&= s_i(l)s_j(l).
\end{aligned}
$$

This completes the proof of Theorem 3.2.2. □

It is known that the character table $\tilde{P}^- = (\tilde{s}_j(i))$ of $\mathfrak{X}(GO_3(q), \Theta_3(q))$ is described as follows (cf. Bannai-Kwok-Song [6, p.139, Remark 1.]):

$$
\tilde{P}^- = \begin{bmatrix} 1 & (q+1) & \ldots & (q+1) \\ 1 & & & \\ \vdots & & (-\chi_{ij})_{2\leq i,j\leq \frac{q}{2}} & \\ 1 & & & \end{bmatrix}. \tag{88}
$$

Thus it follows from Theorem 3.2.2 that the character table of $\mathfrak{X}(GO_{2m+1}(q), \Theta_{2m+1}(q))$ is controlled by that of $\mathfrak{X}(GO_3(q), \Theta_3(q))$, by replacing $q \to q^{m-1}$.

4 Subschemes

4.1 Subschemes of $\mathfrak{X}(GO_{2m+1}(q), \Omega_{2m+1}(q))$

First of all, we prove the following theorem:

Theorem 4.1.1. *$\mathfrak{X}(GO_{2m+1}(q), \Omega_{2m+1}(q))$ is a subscheme of $\mathfrak{X}(GO_3(q^m), \Omega_3(q^m))$.*

The underlying vector space $\mathbb{V}$ is decomposed as

$$\mathbb{V} = \langle e_{11}, e_{21}\rangle \perp \ldots \perp \langle e_{1m}, e_{2m}\rangle \perp \langle r\rangle,$$

where $\{e_{1i}, e_{2i}\}$ $(1 \le i \le m)$ are hyperbolic pairs and as usual $Q(r) = 1$. Let

$$U := \langle e_{11}, e_{21}\rangle \perp \ldots \perp \langle e_{1m}, e_{2m}\rangle$$

be an element in $\Omega_{2m+1}(q)$, then $f|_U$ is a non-degenerate alternating bilinear form on U. The *symplectic group $Sp_{2m}(q)$* is the group of all elements of $GL_{2m}(q) = GL(U)$ which preserve the non-degenerate alternating bilinear form $f|_U$. More precisely,

$$Sp_{2m}(q) := \{\tau \in GL(U) \mid f|_U(\tau(u), \tau(v)) = f|_U(u, v) \text{ for all } u, v \in U\}.$$

It is well known that the orthogonal group $GO_{2m+1}(q)$ is isomorphic to the symplectic group $Sp_{2m}(q)$ for even q, but we review this again in a form convenient for our purpose.

Let $E^{(1)} : \mathbb{V} \longrightarrow U$, $E^{(2)} : \mathbb{V} \longrightarrow \langle r\rangle$ be the orthogonal projections, and define a mapping $\Phi : GO_{2m+1}(q) \longrightarrow Sp_{2m}(q)$ by

$$\Phi(A) := A^{(1)}|_U$$

for $A \in GO_{2m+1}(q)$, where

$$A^{(i)} := E^{(i)}A \ \ (i = 1, 2).$$

Then we have the following:

Proposition 4.1.2. *The mapping $\Phi : GO_{2m+1}(q) \longrightarrow Sp_{2m}(q)$ is well-defined. Moreover, Φ is an isomorphism of $GO_{2m+1}(q)$ onto $Sp_{2m}(q)$.*

Proof. Let A be an element in $GO_{2m+1}(q)$. Then since A does not move the vector r we have $(A^{(1)})^{-1}(0) = \langle r\rangle$, so that

$$\operatorname{rank} A^{(1)}|_U = \dim A^{(1)}U = \dim U - \dim U \cap \langle r\rangle = \dim U = 2m.$$

Thus $\Phi(A)$ is an element in $GL(U)$. Also since $A^{(2)}\mathbb{V}$ is equal to the radical $\langle r\rangle$ of f, we obtain

$$f(A^{(1)}u, A^{(1)}v) = f(A^{(1)}u + A^{(2)}u, A^{(1)}v + A^{(2)}v) = f(Au, Av) = f(u, v),$$

for all $u, v \in U$, which implies that $\Phi(A)$ belongs to $Sp_{2m}(q)$, namely, the mapping Φ is well-defined. This mapping Φ is also a homomorphism. To show this, let A and A' be two elements in $GO_{2m+1}(q)$. Then since $(E^{(1)}AE^{(2)}A')\mathbb{V} = E^{(1)}\langle r\rangle = 0$, we have

$$E^{(1)}AA' = E^{(1)}A(E^{(1)} + E^{(2)})A' = E^{(1)}AE^{(1)}A' + E^{(1)}AE^{(2)}A' = E^{(1)}AE^{(1)}A',$$

so that $\Phi(AA') = \Phi(A)\Phi(A')$.

It remains to show that Φ is a bijection. Suppose $\Phi(A) = id_U$. Then for any vector u in U we have

$$Q(u) = Q(Au) = Q(u + A^{(2)}u) = Q(u) + Q(A^{(2)}u),$$

from which it follows that $A^{(2)}u = 0$, since otherwise $Q(A^{(2)}u)$ cannot be zero by $Q(r) = 1$. Consequently $Au = u$ for all $u \in U$, that is $A|_U = id_U$. This implies $A = id_{\mathbb{V}}$ since $\mathbb{V} = U \perp \langle r\rangle$. Thus Φ is injective. Finally let B be an element in $Sp_{2m}(q)$ and define an element A in $GL(\mathbb{V})$ by

$$\begin{aligned} Ae_{ij} &:= Be_{ij} + \sqrt{Q(Be_{ij})}\, r, \quad \text{for } i = 1, 2 \text{ and } 1 \le j \le m, \\ Ar &:= r. \end{aligned} \tag{89}$$

Then we have $Q(Ae_{ij}) = 0$ for $i = 1, 2$ and $1 \leq j \leq m$, and for any vector $v = \sum_{i,j} \xi_{ij} e_{ij} + \xi r$ in $\mathbb{V}$ we have

$$
\begin{aligned}
Q(Av) &= \sum_{i,j,k,l} \xi_{ij}\xi_{kl} f(Ae_{ij}, Ae_{kl}) + \xi^2 \\
&= \sum_{i,j,k,l} \xi_{ij}\xi_{kl} f(Be_{ij}, Be_{kl}) + \xi^2 \\
&= \sum_{i,j,k,l} \xi_{ij}\xi_{kl} f(e_{ij}, e_{kl}) + \xi^2 \\
&= Q(v),
\end{aligned}
$$

which implies that A is an element in $GO_{2m+1}(q)$, and clearly we have $\Phi(A) = B$. Thus Φ is surjective. This completes the proof of Proposition 4.1.2. □

Let L be the stabilizer of U in $GO_{2m+1}(q)$, then L is isomorphic to $GO^+_{2m}(q)$. From (89) we have the following:

Corollary 4.1.3. *Let B be an element in $Sp_{2m}(q)$. Then $\Phi^{-1}(B)$ is contained in L if and only if $Q(Be_{ij}) = 0$ for all $i = 1, 2$ and $1 \leq j \leq m$.*

Next, let $\mathbb{V}_0$ be a 3-dimensional vector space over $\mathbb{F}_{q^m}$, and let $Q_0 : \mathbb{V}_0 \longrightarrow \mathbb{F}_{q^m}$ be a non-degenerate quadratic form on $\mathbb{V}_0$ with associated alternating bilinear form $f_0 : \mathbb{V}_0 \times \mathbb{V}_0 \longrightarrow \mathbb{F}_{q^m}$. Then $\mathbb{V}_0$ is decomposed as

$$\mathbb{V}_0 = \langle e_1, e_2 \rangle \perp \langle r_0 \rangle$$

where $\{e_1, e_2\}$ is a hyperbolic pair and $Q_0(r_0) = 1$. Let

$$U_0 := \langle e_1, e_2 \rangle$$

be an element in $\Omega_3(q^m)$, then $f_0|_{U_0}$ is a non-degenerate alternating bilinear form on U_0. Seroussi-Lempel [13] proved that for even q there exists a *trace-orthonormal basis* $\{\omega_1, \omega_2, \ldots, \omega_m\}$ of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$, that is,

$$\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\omega_i \omega_j) = \delta_{ij}, \tag{90}$$

where $\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q} : \mathbb{F}_{q^m} \longrightarrow \mathbb{F}_q$ is the trace map from $\mathbb{F}_{q^m}$ onto $\mathbb{F}_q$. Since U and U_0 are both $2m$-dimensional vector space over $\mathbb{F}_q$, we may identify e_{ij} with $\omega_j e_i$ for $i = 1, 2$ and $1 \leq j \leq m$, and U with U_0. Under this identification, $GL_2(q^m)$ is naturally embedded in $GL_{2m}(q)$.

Proposition 4.1.4. *$Sp_2(q^m)$ is a subgroup of $Sp_{2m}(q)$.*

Proof. Let $u = \sum_{i,j} \xi_{ij} e_{ij}$ and $v = \sum_{i,j} \eta_{ij} e_{ij}$ be two vectors in U, and let $\xi_i := \xi_{i1}\omega_1 + \cdots + \xi_{im}\omega_m$ and $\eta_i := \eta_{i1}\omega_1 + \cdots + \eta_{im}\omega_m$ for $i = 1, 2$. Then by (90) we have

$$
\begin{aligned}
\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(f_0|_{U_0}(u, v)) &= \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(f_0|_{U_0}(\xi_1 e_1 + \xi_2 e_2, \eta_1 e_1 + \eta_2 e_2)) \\
&= \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\xi_1\eta_2 + \xi_2\eta_1) \\
&= \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}((\xi_{11}\omega_1 + \cdots + \xi_{1m}\omega_m)(\eta_{21}\omega_1 + \cdots + \eta_{2m}\omega_m) \\
&\qquad + (\xi_{21}\omega_1 + \cdots + \xi_{2m}\omega_m)(\eta_{11}\omega_1 + \cdots + \eta_{1m}\omega_m)) \\
&= \xi_{11}\eta_{21} + \cdots + \xi_{1m}\eta_{2m} + \xi_{21}\eta_{11} + \cdots + \xi_{2m}\eta_{1m} \\
&= f|_U(u, v).
\end{aligned}
$$

Hence any element in $Sp_2(q^m)$ also preserves the alternating form $f|_U$, which proves Proposition 4.1.4. □

It follows immediately from Proposition 4.1.2 and Proposition 4.1.4 that $GO_3(q^m)$ is a subgroup of $GO_{2m+1}(q)$. Furthermore we have the following:

Proposition 4.1.5. *Let L_0 be the stabilizer of U_0 in $GO_3(q^m)$, then $GO_3(q^m) \cap L = L_0$.*

Proof. For any element $B_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in $Sp_2(q^m)$, let B be the corresponding element in $Sp_{2m}(q)$, that is, B is the mapping obtained by regarding B_0 as a linear mapping over $\mathbb{F}_q$. Then for $1 \leq j \leq m$ we have

$$\begin{aligned} Be_{1j} &= B_0(\omega_j e_1) = \alpha\omega_j e_1 + \gamma\omega_j e_2 \\ &= \alpha_{j1}e_{11} + \cdots + \alpha_{jm}e_{1m} + \gamma_{j1}e_{21} + \cdots + \gamma_{jm}e_{2m} \end{aligned}$$

where $\alpha\omega_j = \alpha_{j1}\omega_1 + \cdots + \alpha_{jm}\omega_m$, $\gamma\omega_j = \gamma_{j1}\omega_1 + \cdots + \gamma_{jm}\omega_m$ for some $\alpha_{jk}, \gamma_{jk} \in \mathbb{F}_q$ $(1 \leq k \leq m)$, from which it follows that

$$Q(Be_{1j}) = \alpha_{j1}\gamma_{j1} + \cdots + \alpha_{jm}\gamma_{jm} = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\alpha\gamma\omega_j^2).$$

Similarly for $1 \leq j \leq m$ we have

$$\begin{aligned} Be_{2j} &= B_0(\omega_j e_2) = \beta\omega_j e_1 + \delta\omega_j e_2 \\ &= \beta_{j1}e_{11} + \cdots + \beta_{jm}e_{1m} + \delta_{j1}e_{21} + \cdots + \delta_{jm}e_{2m} \end{aligned}$$

where $\beta\omega_j = \beta_{j1}\omega_1 + \cdots + \beta_{jm}\omega_m$, $\delta\omega_j = \delta_{j1}\omega_1 + \cdots + \delta_{jm}\omega_m$ for some $\beta_{jk}, \delta_{jk} \in \mathbb{F}_q$ $(1 \leq k \leq m)$, from which it follows that

$$Q(Be_{2j}) = \beta_{j1}\delta_{j1} + \cdots + \beta_{jm}\delta_{jm} = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\beta\delta\omega_j^2).$$

If $\alpha\gamma = \beta\delta = 0$ then clearly $\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\alpha\gamma\omega_j^2) = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\beta\delta\omega_j^2) = 0$ for all $1 \leq j \leq m$. The converse is also true. To show this, suppose contrary. Since $\{\omega_1, \omega_2, \ldots, \omega_m\}$ is a basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$, so is $\{\omega_1^2, \omega_2^2, \ldots, \omega_m^2\}$. Thus it follows that for all $\xi \in \mathbb{F}_{q^m}$ we have $\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\xi) = 0$ since $\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}$ is a linear mapping, which is a contradiction. Therefore by Corollary 4.1.3, $\Phi^{-1}(B)$ is contained in L if and only if $\alpha\gamma = \beta\delta = 0$. In the same way as before let $E_0^{(1)} : \mathbb{V}_0 \longrightarrow U_0$, $E_0^{(2)} : \mathbb{V}_0 \longrightarrow \langle r_0 \rangle$ be the orthogonal projections, and define a mapping $\Phi_0 : GO_3(q^m) \longrightarrow Sp_2(q^m)$ by

$$\Phi_0(A_0) := A_0^{(1)}|_{U_0}$$

for $A_0 \in GO_3(q^m)$, where

$$A_0^{(i)} := E_0^{(i)} A_0 \ \ (i = 1, 2).$$

Since actually we chose m arbitrarily, it also follows that $\Phi_0^{-1}(B_0)$ is contained in L_0 if and only if $\alpha\gamma = \beta\delta = 0$, which proves Proposition 4.1.5. ☐

Remark. As is in the proof of Proposition 4.1.5, $GO_2^+(q)$ is isomorphic to

$$\left\{ \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix}, \begin{pmatrix} & z \\ z^{-1} & \end{pmatrix} \;\middle|\; z \in \mathbb{F}_q^* \right\},$$

which is in turn isomorphic to the *dihedral group* $D_{2(q-1)}$ of order $2(q-1)$.

By Proposition 4.1.5 the containment relations among $GO_{2m+1}(q)$, $GO_{2m}^+(q)$, $GO_3(q^m)$ and $GO_2^+(q^m)$ are displayed in the following diagram:

$$\begin{array}{ccc} GO_{2m+1}(q) & \supset & GO_{2m}^+(q) \\ \cup & & \cup \\ GO_3(q^m) & \supset & GO_2^+(q^m) \end{array}$$

where $GO_3(q^m) \cap GO_{2m}^+(q) = GO_2^+(q^m)$.

Proof of Theorem 4.1.1. It follows from the above diagram that each left coset of $GO_{2m+1}(q)$ by $GO_{2m}^+(q)$ contains at most one left coset of $GO_3(q^m)$ by $GO_2^+(q^m)$, since for any two elements A_0, A_0' in $GO_3(q^m)$, we have $A_0^{-1}A_0' \in GO_2^+(q^m)$ if and only if $A_0^{-1}A_0' \in GO_{2m}^+(q)$. Moreover from (2) it follows that

$$|GO_{2m+1}(q) : GO_{2m}^+(q)| = |GO_3(q^m) : GO_2^+(q^m)| = \frac{q^m(q^m+1)}{2},$$

so that each left coset of $GO_{2m+1}(q)$ by $GO^+_{2m}(q)$ contains exactly one left coset of $GO_3(q^m)$ by $GO^+_2(q^m)$. Therefore the action of $GO_3(q^m)$ on $GO_{2m+1}(q)/GO^+_{2m}(q)$ is equivalent to the action on $GO_3(q^m)/GO^+_2(q^m)$, which completes the proof of Theorem 4.1.1. □

From now on, we determine how to merge the relations of $\mathfrak{X}(GO_3(q^m), \Omega_3(q^m))$ to get the subscheme $\mathfrak{X}(GO_{2m+1}(q), \Omega_{2m+1}(q))$. We use the notation in the proof of Proposition 4.1.5, and also we mainly use the symbol " $\tilde{}$ " to stand for $GO_3(q^m)$ case. Namely we let $\tilde{\nu}$ be a primitive element of $\mathbb{F}_{q^m}$, and define

$$\tilde{\lambda}_i := \frac{\tilde{\nu}^{i-1}}{1+\tilde{\nu}^{i-1}} \quad \text{for } 2 \le i \le \tfrac{q^m}{2},$$

and

$$\tilde{\mu}_i := \sqrt{\tilde{\lambda}_i^2 + \tilde{\lambda}_i}.$$

Also we let $\{\tilde{R}_i\}_{0 \le i \le \frac{q^m}{2}}$ denotes the set of relations of $\mathfrak{X}(GO_3(q^m), \Omega_3(q^m))$.

(i) Define a mapping $A_0 : \mathbb{V}_0 \longrightarrow \mathbb{V}_0$ by

$$\begin{aligned} A_0 e_1 &:= e_1 + e_2 + r_0, \\ A_0 e_2 &:= e_2, \\ A_0 r_0 &:= r_0. \end{aligned}$$

Then we have $Q_0(A_0e_1) = Q_0(A_0e_2) = 0$ and $f_0(A_0e_1, A_0e_2) = 1$ so that A_0 is an element in $GO_3(q^m)$. Let $V_0 := A_0U_0 \in \Omega_3(q^m)$, then we have

$$U_0 \cap V_0 = \langle e_2 \rangle^\perp \cap U_0 = \langle e_2 \rangle^\perp \cap V_0,$$

from which it follows that $(U_0, V_0) \in \tilde{R}_1$. By definition, the mapping $B_0 := \Phi_0(A_0) \in Sp_2(q^m)$ is defined by

$$\begin{aligned} B_0 e_1 &= e_1 + e_2, \\ B_0 e_2 &= e_2. \end{aligned}$$

Let B denotes the element in $Sp_{2m}(q)$ corresponding to B_0, then B is given by

$$\begin{aligned} Be_{1j} &= B_0(\omega_j e_1) = \omega_j e_1 + \omega_j e_2 = e_{1j} + e_{2j}, \\ Be_{2j} &= B_0(\omega_j e_2) = \omega_j e_2 = e_{2j}. \end{aligned}$$

Since $Q(Be_{1j}) = 1$ and $Q(Be_{2j}) = 0$, it follows from (89) that $A := \Phi^{-1}(B)$ is obtained as

$$\begin{aligned} Ae_{1j} &= e'_{1j} := e_{1j} + e_{2j} + r \quad \text{for } 1 \le j \le m, \\ Ae_{2j} &= e'_{2j} := e_{2j} \quad \text{for } 1 \le j \le m, \\ Ar &= r. \end{aligned}$$

Let $V := AU$ be an element in $\Omega_{2m+1}(q)$, and define a vector w in $U \cap V$ by

$$w := e_{21} + e_{22} + \cdots + e_{2m} = e'_{21} + e'_{22} + \cdots + e'_{2m}.$$

Then $w \neq 0$ and it follows that

$$U \cap V = \langle w \rangle^\perp \cap U = \langle w \rangle^\perp \cap V.$$

To show this, let $y = \sum_{i,j} \xi_{ij} e'_{ij}$ be a vector in V orthogonal to w, then the r-component of y with respect to the basis $\{e_{ij}\}_{i,j} \cup \{r\}$ is equal to

$$\xi_{11} + \xi_{12} + \cdots + \xi_{1m} = f(w, y) = 0,$$

so that y belongs to $U \cap V$, as desired. Since $Q(w) = 0$ we have $(U, V) \in R_1$. That is, the relation $\tilde{R}_1$ is merged into the relation R_1.

(ii) Next, for $2 \le l \le \frac{q^m}{2}$ define a mapping $A_0 : \mathbb{V}_0 \longrightarrow \mathbb{V}_0$ by

$$\begin{aligned} A_0 e_1 &= e'_1 := (\tilde{\lambda}_l + 1)e_1 + \tilde{\lambda}_l e_2 + \tilde{\mu}_l\, r_0, \\ A_0 e_2 &= e'_2 := \tilde{\lambda}_l e_1 + (\tilde{\lambda}_l + 1)e_2 + \tilde{\mu}_l\, r_0, \\ A_0 r_0 &:= r_0. \end{aligned}$$

Then we have $Q_0(e'_1) = Q_0(e'_2) = 0$ and $f_0(e'_1, e'_2) = 1$ so that A_0 is an element in $GO_3(q^m)$. Let $V_0 := A_0 U_0 \in \Omega_3(q^m)$, and let $w_0 := e_1 + e_2 = e'_1 + e'_2$ be a vector in $U_0 \cap V_0$, then we have

$$U_0 \cap V_0 = \langle w_0 \rangle^\perp \cap U_0 = \langle w_0 \rangle^\perp \cap V_0.$$

Since $Q_0(w_0) = 1$, $f_0(e_1, w_0) = f_0(e'_1, w_0) = 1$ and $e_1 + e'_1 = \tilde{\lambda}_l w_0 + \tilde{\mu}_l\, r_0$, it follows that $(U_0, V_0) \in \tilde{R}_l$. The mapping $B_0 := \Phi_0(A_0) \in Sp_2(q^m)$ is defined by

$$\begin{aligned} B_0 e_1 &= (\tilde{\lambda}_l + 1)e_1 + \tilde{\lambda}_l e_2, \\ B_0 e_2 &= \tilde{\lambda}_l e_1 + (\tilde{\lambda}_l + 1)e_2. \end{aligned}$$

Let

$$\tilde{\lambda}_l \omega_j = \lambda_{lj1}\omega_1 + \cdots + \lambda_{ljm}\omega_m \quad \text{for } 1 \le j \le m,$$

and

$$\tilde{\mu}_l = \mu_{l1}\omega_1 + \cdots + \mu_{lm}\omega_m,$$

for some $\lambda_{ljk}, \mu_{lk} \in \mathbb{F}_q$. Notice that the coefficients λ_{ljk}, μ_{lk} are given by

$$\lambda_{ljk} = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l \omega_j \omega_k) \quad \text{and } \mu_{lk} = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\mu}_l \omega_k), \tag{91}$$

for $1 \le j, k \le m$. Let B be the element in $Sp_{2m}(q)$ corresponding to B_0. Then we have

$$\begin{aligned} Be_{1j} &= (\tilde{\lambda}_l + 1)\omega_j e_1 + \tilde{\lambda}_l \omega_j e_2 \\ &= \lambda_{lj1}(e_{11} + e_{21}) + \cdots + \lambda_{ljm}(e_{1m} + e_{2m}) + e_{1j}, \quad \text{for } 1 \le j \le m, \\ Be_{2j} &= \tilde{\lambda}_l \omega_j e_1 + (\tilde{\lambda}_l + 1)\omega_j e_2 \\ &= \lambda_{lj1}(e_{11} + e_{21}) + \cdots + \lambda_{ljm}(e_{1m} + e_{2m}) + e_{2j}, \quad \text{for } 1 \le j \le m. \end{aligned}$$

Since from (91)

$$\begin{aligned} Q(Be_{1j}) = Q(Be_{2j}) &= \lambda_{lj1}^2 + \cdots + \lambda_{ljm}^2 + \lambda_{ljj} \\ &= \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l^2 \omega_j^2) + \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l \omega_j^2) \\ &= \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\mu}_l^2 \omega_j^2) \\ &= \mu_{lj}^2, \end{aligned}$$

it follows from (89) that the mapping $A := \Phi^{-1}(B) \in GO_{2m+1}(q)$ is given by

$$\begin{aligned} Ae_{1j} &= e'_{1j} := \lambda_{lj1}(e_{11} + e_{21}) + \cdots + \lambda_{ljm}(e_{1m} + e_{2m}) + e_{1j} + \mu_{lj} r, \quad \text{for } 1 \le j \le m, \\ Ae_{2j} &= e'_{2j} := \lambda_{lj1}(e_{11} + e_{21}) + \cdots + \lambda_{ljm}(e_{1m} + e_{2m}) + e_{2j} + \mu_{lj} r, \quad \text{for } 1 \le j \le m, \\ Ar &= r. \end{aligned}$$

Let $V := AU$ be an element in $\Omega_{2m+1}(q)$. Notice that since $\tilde{\mu}_l \ne 0$ the number of μ_{lj} equal to 0 is at most $m - 1$. Define a vector w' in $U \cap V$ by

$$\begin{aligned} w' &:= \mu_{l1}(e_{11} + e_{21}) + \cdots + \mu_{lm}(e_{1m} + e_{2m}) \\ &= \mu_{l1}(e'_{11} + e'_{21}) + \cdots + \mu_{lm}(e'_{1m} + e'_{2m}). \end{aligned}$$

Then $w' \ne 0$ and it follows that

$$U \cap V = \langle w' \rangle^\perp \cap U = \langle w' \rangle^\perp \cap V.$$

To show this, let $y = \sum_{i,j} \xi_{ij} e'_{ij}$ be a vector in V orthogonal to w', then the r-component of y with respect to the basis $\{e_{ij}\}_{i,j} \cup \{r\}$ is equal to

$$(\xi_{11} + \xi_{21})\mu_{l1} + \cdots + (\xi_{1m} + \xi_{2m})\mu_{lm} = f(w', y) = 0,$$

so that y is contained in $U \cap V$, as desired. Also we have

$$Q(w') = \mu_{l1}^2 + \cdots + \mu_{lm}^2 = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\mu}_l^2), \tag{92}$$

from which it follows that $(U, V) \in R_1$ if and only if

$$\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\mu}_l^2) = 0, \tag{93}$$

or equivalently

$$\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l) = 0 \quad \text{or } \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l) = 1. \tag{94}$$

Suppose $\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\mu}_l^2) \neq 0$, so that $(U, V) \notin R_1$. Let

$$\frac{\tilde{\mu}_l}{\tilde{\lambda}_l} = \chi_{l1}\omega_1 + \cdots + \chi_{lm}\omega_m,$$

for $\chi_{lk} \in \mathbb{F}_q$ $(1 \leq k \leq m)$, that is,

$$\chi_{lk} = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}\left(\frac{\tilde{\mu}_l}{\tilde{\lambda}_l}\omega_k\right) \quad (1 \leq k \leq m),$$

and define two vector $u' \in U$ and $v' \in V$ by

$$\begin{aligned} u' &:= \chi_{l1}e_{11} + \cdots + \chi_{lm}e_{1m}, \\ v' &:= \chi_{l1}e'_{11} + \cdots + \chi_{lm}e'_{1m}. \end{aligned}$$

Then we have $Q(u') = Q(v') = 0$, and

$$\begin{aligned} u' + v' &= \chi_{l1}(e_{11} + e'_{11}) + \cdots + \chi_{lm}(e_{1m} + e'_{1m}) \\ &= \sum_{j=1}^{m} \chi_{lj}\left(\sum_{k=1}^{m} \lambda_{ljk}(e_{1k} + e_{2k}) + \mu_{lj} r\right) \\ &= \sum_{k=1}^{m}\left(\sum_{j=1}^{m} \chi_{lj}\lambda_{ljk}\right)(e_{1k} + e_{2k}) + \left(\sum_{j=1}^{m} \chi_{lj}\mu_{lj}\right) r. \end{aligned}$$

Now it follows from (91) that

$$\sum_{j=1}^{m} \chi_{lj}\lambda_{ljk} = \sum_{j=1}^{m} \chi_{lj}\lambda_{lkj} = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}\left(\frac{\tilde{\mu}_l}{\tilde{\lambda}_l}\tilde{\lambda}_l\omega_k\right) = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\mu}_l\omega_k) = \mu_{lk},$$

for $1 \leq k \leq m$. Also

$$\sum_{j=1}^{m} \chi_{lj}\mu_{lj} = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}\left(\frac{\tilde{\mu}_l}{\tilde{\lambda}_l}\tilde{\mu}_l\right) = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l + 1). \tag{95}$$

Hence we have

$$\begin{aligned} u' + v' &= \sum_{k=1}^{m} \mu_{lk}(e_{1k} + e_{2k}) + \big(\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l + 1)\big) r \\ &= w' + \big(\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l + 1)\big) r, \end{aligned} \tag{96}$$

so that

$$U = \langle u', w' \rangle \perp W, \quad V = \langle v', w' \rangle \perp W, \tag{97}$$

where $W := \langle u', w' \rangle^{\perp} \cap U \subset U \cap V$. It also follows from (95) and (96) that

$$f(u', v') = f(u', w') = f(v', w') = \sum_{j=1}^{m} \chi_{lj} \mu_{lj} = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l + 1). \tag{98}$$

Here $\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l + 1) \neq 0$ by assumption. Define

$$w := \frac{1}{\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\mu}_l)} w',$$

and

$$u := \frac{\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\mu}_l)}{\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l + 1)} u', \quad v := \frac{\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\mu}_l)}{\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l + 1)} v'.$$

Then $Q(u) = Q(v) = 0$, and it follows from (92), (98), (97) that $Q(w) = 1$, $f(u, w) = f(v, w) = 1$, and

$$U = \langle u, w \rangle \perp W, \quad V = \langle v, w \rangle \perp W.$$

Also by (98) we have

$$\begin{aligned} f(u, v) &= \frac{\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\mu}_l^2)}{\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l + 1)} = \frac{(\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l))(\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l) + 1)}{\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l + 1)} \\ &= \begin{cases} \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l) & \text{if } m\text{: odd,} \\ \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l) + 1 & \text{if } m\text{: even.} \end{cases} \end{aligned}$$

Thus (U, V) belongs to R_k for some $k \in \{2, 3, \ldots, \frac{q}{2}\}$ such that

$$\lambda_k = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l), \quad \text{or } \lambda_k = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l) + 1, \tag{99}$$

which is equivalent to

$$\mu_k = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\mu}_l). \tag{100}$$

To summarize we have the following:

Proposition 4.1.6. *Define* $(\frac{q}{2} - 1)$ *relations* $R_1, R_2, \ldots, R_{\frac{q}{2}}$ *on* $\Omega_3(q^m)$ *by*

$$R_j := \bigcup_{i \in \Xi_j} \tilde{R}_i \quad (1 \leq j \leq \tfrac{q}{2}),$$

where

$$\begin{aligned} \Xi_1 &:= \left\{ i \in \{2, 3, \ldots, \tfrac{q}{2}\} \middle| \, \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\mu}_i) = 0 \right\} \cup \{1\}, \\ \Xi_j &:= \left\{ i \in \{2, 3, \ldots, \tfrac{q}{2}\} \middle| \, \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\mu}_i) = \mu_j \right\} \quad (2 \leq j \leq \tfrac{q}{2}). \end{aligned}$$

Then these $(\frac{q}{2} - 1)$ *relations, together with* $R_0 := \tilde{R}_0$*, form the subscheme of* $\mathfrak{X}(GO_3(q^m), \Omega_3(q^m))$ *isomorphic to* $\mathfrak{X}(GO_{2m+1}(q), \Omega_{2m+1}(q))$.

Corollary 4.1.7. $\mathfrak{X}(GO_{2n+1}(q), \Omega_{2n+1}(q))$ *is a subscheme of* $\mathfrak{X}(GO_{2m+1}(q^{\frac{n}{m}}), \Omega_{2m+1}(q^{\frac{n}{m}}))$ *whenever* m *devides* n.

Proof. This is an immediate consequence of Proposition 4.1.6 and Lemma 4.1.8 below (cf. Lidl-Niederreiter [10, p.56, Theorem 2.26]). In fact, these two association schemes are both subschemes of $\mathfrak{X}(GO_3(q^n), \Omega_3(q^n))$ by Theorem 4.1.1. □

Lemma 4.1.8. *If* m *devides* n*, then*

$$\mathrm{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q} = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q} \circ \mathrm{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_{q^m}}.$$

4.2 Subschemes of $\mathfrak{X}(GO_{2m+1}(q), \Theta_{2m+1}(q))$

First of all, we prove the following theorem.

Theorem 4.2.1. *$\mathfrak{X}(GO_{2m+1}(q), \Theta_{2m+1}(q))$ is a subscheme of $\mathfrak{X}(GO_3(q^m), \Theta_3(q^m))$.*

Let $t^2 + t + \pi$ be an irreducible polynomial over $\mathbb{F}_q$. Then the underlying vector space $\mathbb{V}$ is decomposed as

$$\mathbb{V} = \langle e_{11}, e_{21}\rangle \perp \ldots \perp \langle e_{1m}, e_{2m}\rangle \perp \langle r\rangle,$$

where $\{e_{1i}, e_{2i}\}$ $(1 \le i \le m-1)$ are hyperbolic pairs, $Q(e_{1m}) = 1$, $Q(e_{2m}) = \pi$, $f(e_{1m}, e_{2m}) = 1$ and as usual $Q(r) = 1$. Let

$$U := \langle e_{11}, e_{21}\rangle \perp \ldots \perp \langle e_{1m}, e_{2m}\rangle$$

be an element in $\Theta_{2m+1}(q)$, then $f|_U$ is a non-degenerate alternating bilinear form on U. This time, we consider the symplectic group $Sp_{2m}(q)$ with respect to $f|_U$, that is,

$$Sp_{2m}(q) := \{\tau \in GL(U) \mid f|_U(\tau(u), \tau(v)) = f|_U(u, v) \text{ for all } u, v \in U\}.$$

Let $E^{(1)} : \mathbb{V} \longrightarrow U$, $E^{(2)} : \mathbb{V} \longrightarrow \langle r\rangle$ be the orthogonal projections, and define a mapping $\Psi : GO_{2m+1}(q) \longrightarrow Sp_{2m}(q)$ by

$$\Psi(A) := A^{(1)}|_U$$

for $A \in GO_{2m+1}(q)$, where

$$A^{(i)} := E^{(i)}A \ \ (i = 1, 2).$$

Then we have the following:

Proposition 4.2.2. *The mapping $\Psi : GO_{2m+1}(q) \longrightarrow Sp_{2m}(q)$ is well-defined. Moreover, Ψ is an isomorphism of $GO_{2m+1}(q)$ onto $Sp_{2m}(q)$.*

Proof. Let A be an element in $GO_{2m+1}(q)$. Then since A does not move the vector r we have $(A^{(1)})^{-1}(0) = \langle r\rangle$, so that

$$\operatorname{rank} A^{(1)}|_U = \dim A^{(1)}U = \dim U - \dim U \cap \langle r\rangle = \dim U = 2m.$$

Thus $\Psi(A)$ is an element in $GL(U)$. Also since $A^{(2)}\mathbb{V}$ is equal to the radical $\langle r\rangle$ of f, we obtain

$$f(A^{(1)}u, A^{(1)}v) = f(A^{(1)}u + A^{(2)}u, A^{(1)}v + A^{(2)}v) = f(Au, Av) = f(u, v),$$

for all $u, v \in U$, which implies that $\Psi(A)$ belongs to $Sp_{2m}(q)$, namely, the mapping Ψ is well-defined. This mapping Ψ is also a homomorphism. To show this, let A and A' be two elements in $GO_{2m+1}(q)$. Then since $(E^{(1)}AE^{(2)}A')\mathbb{V} = E^{(1)}\langle r\rangle = 0$, we have

$$E^{(1)}AA' = E^{(1)}A(E^{(1)} + E^{(2)})A' = E^{(1)}AE^{(1)}A' + E^{(1)}AE^{(2)}A' = E^{(1)}AE^{(1)}A',$$

so that $\Psi(AA') = \Psi(A)\Psi(A')$.

It remains to show that Ψ is a bijection. Suppose $\Psi(A) = id_U$. Then for any vector u in U we have

$$Q(u) = Q(Au) = Q(u + A^{(2)}u) = Q(u) + Q(A^{(2)}u),$$

from which it follows that $A^{(2)}u = 0$, since otherwise $Q(A^{(2)}u)$ cannot be zero by $Q(r) = 1$. Consequently $Au = u$ for all $u \in U$, that is $A|_U = id_U$. This implies $A = id_{\mathbb{V}}$ since $\mathbb{V} = U \perp \langle r\rangle$. Thus Ψ is injective. Finally let B be an element in $Sp_{2m}(q)$ and define an element A in $GL(\mathbb{V})$ by

$$\begin{aligned} Ae_{ij} &:= Be_{ij} + \sqrt{Q(Be_{ij})}\, r, \quad \text{for } i = 1, 2 \text{ and } 1 \le j \le m-1, \\ Ae_{1m} &:= Be_{1m} + (\sqrt{Q(Be_{1m})} + 1)r, \\ Ae_{2m} &:= Be_{2m} + (\sqrt{Q(Be_{2m})} + \sqrt{\pi})r, \\ Ar &:= r. \end{aligned} \tag{101}$$

Then we have $Q(Ae_{ij}) = 0$ for $i = 1, 2$ and $1 \leq j \leq m-1$, $Q(Ae_{1m}) = 1$, $Q(Ae_{2m}) = \pi$ and for any vector $v = \sum_{i,j} \xi_{ij} e_{ij} + \xi r$ in $\mathbb{V}$ we have

$$\begin{aligned} Q(Av) &= \sum_{i,j,k,l} \xi_{ij}\xi_{kl} f(Ae_{ij}, Ae_{kl}) + \xi_{1m}^2 + \pi\xi_{2m}^2 + \xi^2 \\ &= \sum_{i,j,k,l} \xi_{ij}\xi_{kl} f(Be_{ij}, Be_{kl}) + \xi_{1m}^2 + \pi\xi_{2m}^2 + \xi^2 \\ &= \sum_{i,j,k,l} \xi_{ij}\xi_{kl} f(e_{ij}, e_{kl}) + \xi_{1m}^2 + \pi\xi_{2m}^2 + \xi^2 \\ &= Q(v), \end{aligned}$$

which implies that A is an element in $GO_{2m+1}(q)$, and clearly we have $\Psi(A) = B$. Thus Ψ is surjective. This completes the proof of Proposition 4.2.2. □

Let L be the stabilizer of U in $GO_{2m+1}(q)$, then L is isomorphic to $GO_{2m}^-(q)$. From (101) we have the following:

Corollary 4.2.3. *Let B be an element in $Sp_{2m}(q)$. Then $\Psi^{-1}(B)$ is contained in L if and only if $Q(Be_{ij}) = 0$ for all $i = 1, 2$ and $1 \leq j \leq m-1$, $Q(Be_{1m}) = 1$ and $Q(Be_{2m}) = \pi$.*

Next, let $\mathbb{V}_0$ be a 3-dimensional vector space over $\mathbb{F}_{q^m}$, and let $Q_0 : \mathbb{V}_0 \longrightarrow \mathbb{F}_{q^m}$ be a non-degenerate quadratic form on $\mathbb{V}_0$ with associated alternating bilinear form $f_0 : \mathbb{V}_0 \times \mathbb{V}_0 \longrightarrow \mathbb{F}_{q^m}$. As mentioned before, there exists a trace-orthonormal basis $\{\omega_1, \omega_2, \ldots, \omega_m\}$ of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$, that is,

$$\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\omega_i \omega_j) = \delta_{ij}, \tag{102}$$

where $\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q} : \mathbb{F}_{q^m} \longrightarrow \mathbb{F}_q$ is the trace map from $\mathbb{F}_{q^m}$ onto $\mathbb{F}_q$ (cf. (90)). Then the polynomial $\omega_m^2 t^2 + t + \pi\omega_m^2 \in \mathbb{F}_{q^m}[t]$ is irreducible over $\mathbb{F}_{q^m}$. In order to show this, we make use of the following lemma (cf. Lidl-Niederreiter [10, p.56, Theorem 2.25]):

Lemma 4.2.4. *A polynomial $t^2 + t + \alpha$ in $\mathbb{F}_q[t]$ is irreducible over $\mathbb{F}_q$ if and only if $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\alpha) = 1$.*

Since $t^2 + t + \pi$ is irreducible over $\mathbb{F}_q$, it follows from Lemma 4.1.8 that

$$\begin{aligned} \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_2}(\pi\omega_m^4) &= \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\pi\omega_m^4)) \\ &= \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\pi(\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\omega_m^2))^2) \\ &= \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\pi) \\ &= 1, \end{aligned}$$

so that $(t')^2 + t' + \pi\omega_m^4 \in \mathbb{F}_{q^m}[t']$ is an irreducible polynomial over $\mathbb{F}_{q^m}$. By putting $t' := \omega_m^2 t$, this also implies that $\omega_m^2 t^2 + t + \pi\omega_m^2 \in \mathbb{F}_{q^m}[t]$ is irreducible over $\mathbb{F}_{q^m}$, as desired.

Therefore $\mathbb{V}_0$ is decomposed as

$$\mathbb{V}_0 = \langle e_1, e_2 \rangle \perp \langle r_0 \rangle$$

where $Q_0(e_1) = \omega_m^2$, $Q_0(e_2) = \pi\omega_m^2$, $f_0(e_1, e_2) = 1$ and $Q_0(r_0) = 1$. Let

$$U_0 := \langle e_1, e_2 \rangle$$

be an element in $\Theta_3(q^m)$, then $f_0|_{U_0}$ is a non-degenerate alternating bilinear form on U_0. Since U and U_0 are both $2m$-dimensional vector space over $\mathbb{F}_q$, we may identify e_{ij} with $\omega_j e_i$ for $i = 1, 2$ and $1 \leq j \leq m$, and U with U_0. Under this identification, $GL_2(q^m)$ is naturally embedded in $GL_{2m}(q)$.

Proposition 4.2.5. *$Sp_2(q^m)$ is a subgroup of $Sp_{2m}(q)$.*

Proof. Let $u = \sum_{i,j} \xi_{ij} e_{ij}$ and $v = \sum_{i,j} \eta_{ij} e_{ij}$ be two vectors in U, and let $\xi_i := \xi_{i1}\omega_1 + \cdots + \xi_{im}\omega_m$ and $\eta_i := \eta_{i1}\omega_1 + \cdots + \eta_{im}\omega_m$ for $i = 1, 2$. Then by (102) we have

$$\begin{aligned}
\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(f_0|_{U_0}(u, v)) &= \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(f_0|_{U_0}(\xi_1 e_1 + \xi_2 e_2, \eta_1 e_1 + \eta_2 e_2)) \\
&= \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\xi_1 \eta_2 + \xi_2 \eta_1) \\
&= \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}((\xi_{11}\omega_1 + \cdots + \xi_{1m}\omega_m)(\eta_{21}\omega_1 + \cdots + \eta_{2m}\omega_m) \\
&\qquad\qquad + (\xi_{21}\omega_1 + \cdots + \xi_{2m}\omega_m)(\eta_{11}\omega_1 + \cdots + \eta_{1m}\omega_m)) \\
&= \xi_{11}\eta_{21} + \cdots + \xi_{1m}\eta_{2m} + \xi_{21}\eta_{11} + \cdots + \xi_{2m}\eta_{1m} \\
&= f|_U(u, v).
\end{aligned}$$

Hence any element in $Sp_2(q^m)$ also preserves the alternating form $f|_U$, which proves Proposition 4.2.5. □

It follows immediately from Proposition 4.2.2 and Proposition 4.2.5 that $GO_3(q^m)$ is a subgroup of $GO_{2m+1}(q)$. Furthermore we have the following:

Proposition 4.2.6. *Let L_0 be the stabilizer of U_0 in $GO_3(q^m)$, then $GO_3(q^m) \cap L = L_0$.*

Proof. For any element $B_0 = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ in $Sp_2(q^m)$, let B be the corresponding element in $Sp_{2m}(q)$, that is, B is the mapping obtained by regarding B_0 as a linear mapping over $\mathbb{F}_q$. Then for $1 \le j \le m$ we have

$$\begin{aligned}
Be_{1j} &= B_0(\omega_j e_1) = \alpha\omega_j e_1 + \gamma\omega_j e_2 \\
&= \alpha_{j1} e_{11} + \cdots + \alpha_{jm} e_{1m} + \gamma_{j1} e_{21} + \cdots + \gamma_{jm} e_{2m}
\end{aligned}$$

where $\alpha\omega_j = \alpha_{j1}\omega_1 + \cdots + \alpha_{jm}\omega_m$, $\gamma\omega_j = \gamma_{j1}\omega_1 + \cdots + \gamma_{jm}\omega_m$ for some $\alpha_{jk}, \gamma_{jk} \in \mathbb{F}_q$ $(1 \le k \le m)$, from which it follows that

$$\begin{aligned}
Q(Be_{1j}) &= \alpha_{j1}\gamma_{j1} + \cdots + \alpha_{jm}\gamma_{jm} + \alpha_{jm}^2 + \pi\gamma_{jm}^2 \\
&= \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\alpha\gamma\omega_j^2) + (\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\alpha\omega_j\omega_m))^2 + \pi(\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\gamma\omega_j\omega_m))^2 \\
&= \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}((\alpha^2\omega_m^2 + \alpha\gamma + \pi\gamma^2\omega_m^2)\omega_j^2).
\end{aligned}$$

Similarly for $1 \le j \le m$ we have

$$\begin{aligned}
Be_{2j} &= B_0(\omega_j e_2) = \beta\omega_j e_1 + \delta\omega_j e_2 \\
&= \beta_{j1} e_{11} + \cdots + \beta_{jm} e_{1m} + \delta_{j1} e_{21} + \cdots + \delta_{jm} e_{2m}
\end{aligned}$$

where $\beta\omega_j = \beta_{j1}\omega_1 + \cdots + \beta_{jm}\omega_m$, $\delta\omega_j = \delta_{j1}\omega_1 + \cdots + \delta_{jm}\omega_m$ for some $\beta_{jk}, \delta_{jk} \in \mathbb{F}_q$ $(1 \le k \le m)$, from which it follows that

$$\begin{aligned}
Q(Be_{2j}) &= \beta_{j1}\delta_{j1} + \cdots + \beta_{jm}\delta_{jm} + \beta_{jm}^2 + \pi\delta_{jm}^2 \\
&= \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\beta\delta\omega_j^2) + (\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\beta\omega_j\omega_m))^2 + \pi(\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\delta\omega_j\omega_m))^2 \\
&= \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}((\beta^2\omega_m^2 + \beta\delta + \pi\delta^2\omega_m^2)\omega_j^2).
\end{aligned}$$

Therefore by Corollary 4.2.3, $\Psi^{-1}(B)$ is contained in L if and only if

$$\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}((\alpha^2\omega_m^2 + \alpha\gamma + \pi\gamma^2\omega_m^2)\omega_j^2) = \begin{cases} 1 & \text{if } j = m, \\ 0 & \text{if } j \ne m, \end{cases}$$

and

$$\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}((\beta^2\omega_m^2 + \beta\delta + \pi\delta^2\omega_m^2)\omega_j^2) = \begin{cases} \pi & \text{if } j = m, \\ 0 & \text{if } j \ne m, \end{cases}$$

which is equivalent to

$$\alpha^2\omega_m^2 + \alpha\gamma + \pi\gamma^2\omega_m^2 = \omega_m^2, \tag{103}$$

and

$$\beta^2\omega_m^2 + \beta\delta + \pi\delta^2\omega_m^2 = \pi\omega_m^2, \tag{104}$$

since $\{\omega_1^2, \omega_2^2, \ldots, \omega_m^2\}$ is also a trace-orthonormal basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. In the same way as before let $E_0^{(1)} : \mathbb{V}_0 \longrightarrow U_0$, $E_0^{(2)} : \mathbb{V}_0 \longrightarrow \langle r_0 \rangle$ be the orthogonal projections, and define a mapping $\Psi_0 : GO_3(q^m) \longrightarrow Sp_2(q^m)$ by

$$\Psi_0(A_0) := A_0^{(1)}|_{U_0}$$

for $A_0 \in GO_3(q^m)$, where

$$A_0^{(i)} := E_0^{(i)} A_0 \;\; (i = 1, 2).$$

Then in this case $A_0 = \Psi_0^{-1}(B_0)$ is given by

$$\begin{aligned} A_0 e_1 &:= B_0 e_1 + (\sqrt{Q_0(B_0 e_1)} + \omega_m) r_0, \\ A_0 e_2 &:= B_0 e_2 + (\sqrt{Q_0(B_0 e_2)} + \sqrt{\pi}\,\omega_m) r_0, \\ A_0 r_0 &:= r_0. \end{aligned}$$

Thus it follows that $A_0 = \Psi_0^{-1}(B_0)$ is contained in L_0 if and only if (103) and (104) are satisfied, which proves Proposition 4.2.6. □

By Proposition 4.2.6 the containment relations among $GO_{2m+1}(q)$, $GO_{2m}^-(q)$, $GO_3(q^m)$ and $GO_2^-(q^m)$ are displayed in the following diagram:

$$\begin{array}{ccc} GO_{2m+1}(q) & \supset & GO_{2m}^-(q) \\ \cup & & \cup \\ GO_3(q^m) & \supset & GO_2^-(q^m) \end{array}$$

where $GO_3(q^m) \cap GO_{2m}^-(q) = GO_2^-(q^m)$.

Proof of Theorem 4.2.1. It follows from the above diagram that each left coset of $GO_{2m+1}(q)$ by $GO_{2m}^-(q)$ contains at most one left coset of $GO_3(q^m)$ by $GO_2^-(q^m)$, since for any two elements A_0, A_0' in $GO_3(q^m)$, we have $A_0^{-1}A_0' \in GO_2^-(q^m)$ if and only if $A_0^{-1}A_0' \in GO_{2m}^-(q)$. Moreover from (3) it follows that

$$|GO_{2m+1}(q) : GO_{2m}^-(q)| = |GO_3(q^m) : GO_2^-(q^m)| = \frac{q^m(q^m-1)}{2},$$

so that each left coset of $GO_{2m+1}(q)$ by $GO_{2m}^-(q)$ contains exactly one left coset of $GO_3(q^m)$ by $GO_2^-(q^m)$. Therefore the action of $GO_3(q^m)$ on $GO_{2m+1}(q)/GO_{2m}^-(q)$ is equivalent to the action on $GO_3(q^m)/GO_2^-(q^m)$, which completes the proof of Theorem 4.2.1. □

From now on, we determine how to merge the relations of $\mathfrak{X}(GO_3(q^m), \Theta_3(q^m))$ to get the subscheme $\mathfrak{X}(GO_{2m+1}(q), \Theta_{2m+1}(q))$. We use the notation in the proof of Proposition 4.2.6, and in the same manner as previous subsection, we mainly use the symbol " ˜ " to stand for $GO_3(q^m)$ case. Namely we let $\tilde{\nu}$ be a primitive element of $\mathbb{F}_{q^m}$, and define

$$\tilde{\lambda}_i := \frac{\tilde{\nu}^{i-1}}{1+\tilde{\nu}^{i-1}} \quad \text{for } 2 \leq i \leq \tfrac{q^m}{2},$$

and

$$\tilde{\mu}_i := \sqrt{\tilde{\lambda}_i^2 + \tilde{\lambda}_i}.$$

Also we let $\tilde{S}_0, \tilde{S}_2, \tilde{S}_3, \ldots, \tilde{S}_{\frac{q^m}{2}}$ denotes the relations of $\mathfrak{X}(GO_3(q^m), \Theta_3(q^m))$.

For $2 \leq l \leq \frac{q^m}{2}$ define a mapping $A_0 : \mathbb{V}_0 \longrightarrow \mathbb{V}_0$ by

$$\begin{aligned}
A_0 e_1 &= e_1' := e_1, \\
A_0 e_2 &= e_2' := \frac{\tilde{\lambda}_l}{\omega_m^2} e_1 + e_2 + \frac{\tilde{\mu}_l}{\omega_m} r_0, \\
A_0 r_0 &:= r_0.
\end{aligned}$$

Then we have $Q_0(e_1') = \omega_m^2$, $Q_0(e_2') = \pi\omega_m^2$ and $f_0(e_1', e_2') = 1$ so that A_0 is an element in $GO_3(q^m)$. Let $V_0 := A_0 U_0 \in \Theta_3(q^m)$, and define three vectors w_0, u_0, v_0 by

$$w_0 := \frac{1}{\omega_m} e_1 = \frac{1}{\omega_m} e_1', \quad u_0 := \omega_m e_2, \quad v_0 := \omega_m e_2',$$

then we have $Q_0(w_0) = 1$, $Q_0(u_0) = Q_0(v_0)$, $f_0(u_0, w_0) = f_0(v_0, w_0) = 1$, and $f_0(u_0, v_0) = \tilde{\lambda}_l$. Hence it follows that $(U_0, V_0) \in \tilde{S}_l$. The mapping $B_0 := \Psi_0(A_0) \in Sp_2(q^m)$ is defined by

$$\begin{aligned}
B_0 e_1 &= e_1, \\
B_0 e_2 &= \frac{\tilde{\lambda}_l}{\omega_m^2} e_1 + e_2.
\end{aligned}$$

Let

$$\frac{\tilde{\lambda}_l \omega_j}{\omega_m^2} = \lambda_{lj1}' \omega_1 + \cdots + \lambda_{ljm}' \omega_m \quad \text{for } 1 \leq j \leq m,$$

and

$$\frac{\tilde{\mu}_l}{\omega_m} = \mu_{l1}' \omega_1 + \cdots + \mu_{lm}' \omega_m,$$

for some $\lambda_{ljk}', \mu_{lk}' \in \mathbb{F}_q$. Notice that the coefficients $\lambda_{ljk}', \mu_{lk}'$ are given by

$$\lambda_{ljk}' = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}\left(\frac{\tilde{\lambda}_l \omega_j}{\omega_m^2} \omega_k\right) \quad \text{and } \mu_{lk}' = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}\left(\frac{\tilde{\mu}_l}{\omega_m} \omega_k\right), \tag{105}$$

for $1 \leq j, k \leq m$. Let B be the element in $Sp_{2m}(q)$ corresponding to B_0. Then we have

$$\begin{aligned}
B e_{1j} &= \omega_j e_1 = e_{1j}, \quad \text{for } 1 \leq j \leq m, \\
B e_{2j} &= \frac{\tilde{\lambda}_l}{\omega_m^2} \omega_j e_1 + \omega_j e_2 \\
&= \lambda_{lj1}' e_{11} + \cdots + \lambda_{ljm}' e_{1m} + e_{2j}, \quad \text{for } 1 \leq j \leq m.
\end{aligned}$$

Since from (105)

$$\begin{aligned}
Q(B e_{2j}) &= (\lambda_{ljm}')^2 + \lambda_{ljj}' \\
&= \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}\left(\frac{\tilde{\lambda}_l^2 \omega_j^2}{\omega_m^2}\right) + \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}\left(\frac{\tilde{\lambda}_l \omega_j^2}{\omega_m^2}\right) \\
&= \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}\left(\frac{\tilde{\mu}_l^2 \omega_j^2}{\omega_m^2}\right) \\
&= (\mu_{lj}')^2,
\end{aligned}$$

for $1 \leq j \leq m-1$, and

$$\begin{aligned}
Q(B e_{2m}) &= (\lambda_{lmm}')^2 + \lambda_{lmm}' + \pi \\
&= (\mu_{lm}')^2 + \pi,
\end{aligned}$$

it follows from (101) that the mapping $A := \Psi^{-1}(B) \in GO_{2m+1}(q)$ is given by

$$\begin{aligned}
Ae_{1j} &= e'_{1j} := e_{1j}, \quad \text{for } 1 \leq j \leq m, \\
Ae_{2j} &= e'_{2j} := \lambda'_{lj1}e_{11} + \cdots + \lambda'_{ljm}e_{1m} + e_{2j} + \mu'_{lj}r, \quad \text{for } 1 \leq j \leq m, \\
Ar &= r.
\end{aligned}$$

Let $V := AU$ be an element in $\Theta_{2m+1}(q)$. Notice that since $\tilde{\mu}_l \neq 0$ the number of μ'_{lj} equal to 0 is at most $m-1$. Define a vector w' in $U \cap V$ by

$$\begin{aligned}
w' &:= \mu'_{l1}e_{11} + \cdots + \mu'_{lm}e_{1m} \\
&= \mu'_{l1}e'_{11} + \cdots + \mu'_{lm}e'_{1m},
\end{aligned}$$

then $w' \neq 0$ and it follows that

$$U \cap V = \langle w' \rangle^{\perp} \cap U = \langle w' \rangle^{\perp} \cap V.$$

To show this, let $y = \sum_{i,j} \xi_{ij}e'_{ij}$ be a vector in V orthogonal to w', then the r-component of y with respect to the basis $\{e_{ij}\}_{i,j} \cup \{r\}$ is equal to

$$\xi_{21}\mu'_{l1} + \cdots + \xi_{2m}\mu'_{lm} = f(w', y) = 0,$$

so that y is contained in $U \cap V$, as desired. Also by (105) we have

$$Q(w') = (\mu'_{lm})^2 = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\mu}_l^2), \tag{106}$$

from which it follows that $(U, V) \in S_1$ if and only if

$$\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\mu}_l^2) = 0, \tag{107}$$

or equivalently

$$\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l) = 0 \quad \text{or} \quad \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l) = 1. \tag{108}$$

Suppose $\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\mu}_l^2) \neq 0$, so that $(U, V) \notin S_1$. Let

$$\frac{\tilde{\mu}_l\omega_m}{\tilde{\lambda}_l} = \chi'_{l1}\omega_1 + \cdots + \chi'_{lm}\omega_m,$$

for $\chi'_{lk} \in \mathbb{F}_q$ $(1 \leq k \leq m)$, that is,

$$\chi'_{lk} = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}\left(\frac{\tilde{\mu}_l\omega_m}{\tilde{\lambda}_l}\omega_k\right) \quad (1 \leq k \leq m),$$

and define two vector $u' \in U$ and $v' \in V$ by

$$\begin{aligned}
u' &:= \chi'_{l1}e_{21} + \cdots + \chi'_{lm}e_{2m}, \\
v' &:= \chi'_{l1}e'_{21} + \cdots + \chi'_{lm}e'_{2m}.
\end{aligned}$$

Then we have $Q(u') = Q(v') = \pi(\chi'_{lm})^2$, and

$$\begin{aligned}
u' + v' &= \chi'_{l1}(e_{21} + e'_{21}) + \cdots + \chi'_{lm}(e_{2m} + e'_{2m}) \\
&= \sum_{j=1}^{m} \chi'_{lj}\left(\sum_{k=1}^{m} \lambda'_{ljk}e_{1k} + \mu'_{lj}r\right) \\
&= \sum_{k=1}^{m}\left(\sum_{j=1}^{m} \chi'_{lj}\lambda'_{ljk}\right)e_{1k} + \left(\sum_{j=1}^{m} \chi'_{lj}\mu'_{lj}\right)r.
\end{aligned}$$

Now it follows from (105) that

$$\sum_{j=1}^{m} \chi'_{lj}\lambda'_{ljk} = \sum_{j=1}^{m} \chi'_{lj}\lambda'_{lkj} = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}\left(\frac{\tilde{\mu}_l\omega_m}{\tilde{\lambda}_l}\frac{\tilde{\lambda}_l\omega_k}{\omega_m^2}\right) = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}\left(\frac{\tilde{\mu}_l\omega_k}{\omega_m}\right) = \mu'_{lk},$$

for $1 \le k \le m$. Also

$$\sum_{j=1}^{m} \chi'_{lj}\mu'_{lj} = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}\Big(\frac{\tilde{\mu}_l\omega_m}{\tilde{\lambda}_l}\frac{\tilde{\mu}_l}{\omega_m}\Big) = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}\Big(\frac{\tilde{\mu}_l}{\tilde{\lambda}_l}\tilde{\mu}_l\Big) = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l + 1). \tag{109}$$

Hence we have

$$\begin{aligned} u' + v' &= \sum_{k=1}^{m} \mu'_{lk} e_{1k} + \big(\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l + 1)\big) r \\ &= w' + \big(\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l + 1)\big) r, \end{aligned} \tag{110}$$

so that

$$U = \langle u', w'\rangle \perp W, \quad V = \langle v', w'\rangle \perp W, \tag{111}$$

where $W := \langle u', w'\rangle^{\perp} \cap U \subset U \cap V$. It also follows from (109) and (110) that

$$f(u', v') = f(u', w') = f(v', w') = \sum_{j=1}^{m} \chi'_{lj}\mu'_{lj} = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l + 1). \tag{112}$$

Here $\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l + 1) \neq 0$ by assumption. Define

$$w := \frac{1}{\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\mu}_l)} w',$$

and

$$u := \frac{\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\mu}_l)}{\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l + 1)} u', \quad v := \frac{\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\mu}_l)}{\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l + 1)} v'.$$

Then $Q(u) = Q(v)$, and it follows from (106), (112), (111) that $Q(w) = 1$, $f(u, w) = f(v, w) = 1$, and

$$U = \langle u, w\rangle \perp W, \quad V = \langle v, w\rangle \perp W.$$

Also by (112) we have

$$\begin{aligned} f(u, v) &= \frac{\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\mu}_l^2)}{\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l + 1)} = \frac{(\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l))(\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l) + 1)}{\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l + 1)} \\ &= \begin{cases} \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l) & \text{if } m\text{: odd}, \\ \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l) + 1 & \text{if } m\text{: even}. \end{cases} \end{aligned}$$

Thus (U, V) belongs to S_k for some $k \in \{2, 3, \ldots, \frac{q}{2}\}$ such that

$$\lambda_k = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l), \quad \text{or } \lambda_k = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\lambda}_l) + 1, \tag{113}$$

which is equivalent to

$$\mu_k = \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\mu}_l). \tag{114}$$

To summarize we have the following:

Proposition 4.2.7. *Define $(\frac{q}{2} - 1)$ relations $S_1, S_2, \ldots, S_{\frac{q}{2}}$ on $\Theta_3(q^m)$ by*

$$S_j := \bigcup_{i \in \Xi'_j} \tilde{S}_i \quad (1 \le j \le \tfrac{q}{2}),$$

where

$$\begin{aligned} \Xi'_1 &:= \big\{ i \in \{2, 3, \ldots, \tfrac{q}{2}\} \big| \, \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\mu}_i) = 0 \big\}, \\ \Xi'_j &:= \big\{ i \in \{2, 3, \ldots, \tfrac{q}{2}\} \big| \, \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tilde{\mu}_i) = \mu_j \big\} \quad (2 \le j \le \tfrac{q}{2}). \end{aligned}$$

Then these $(\frac{q}{2} - 1)$ relations, together with $S_0 := \tilde{S}_0$, form the subscheme of $\mathfrak{X}(GO_3(q^m), \Theta_3(q^m))$ isomorphic to $\mathfrak{X}(GO_{2m+1}(q), \Theta_{2m+1}(q))$.

Corollary 4.2.8. *$\mathfrak{X}(GO_{2n+1}(q), \Theta_{2n+1}(q))$ is a subscheme of $\mathfrak{X}(GO_{2m+1}(q^{\frac{n}{m}}), \Theta_{2m+1}(q^{\frac{n}{m}}))$ whenever m devides n.*

Proof. This is an immediate consequence of Proposition 4.2.7 and Lemma 4.1.8. In fact, these two association schemes are both subschemes of $\mathfrak{X}(GO_3(q^n), \Theta_3(q^n))$ by Theorem 4.2.1. □

5 Remarks

Remark 1. The association scheme $\mathfrak{X}(GO_3(q), \Theta_3(q))$ (for even q) is a *quotient association scheme* (cf. Bannai-Ito [4, §2.9]) of the association scheme $\mathfrak{X}(GL_2(q), GL_2(q)/GL_1(q^2))$ which is defined by the action of the general linear group $GL_2(q)$ on the *finite upper half plane* $\mathbb{H}_q = \mathbb{F}_{q^2} \backslash \mathbb{F}_q$. Terras [15] gives details on the property of the finite upper half plane. The original motivation of this research, which was proposed by Professor E. Bannai, was to find a connection between two association schemes $\mathfrak{X}(GO_{2m+1}(q), \Theta_{2m+1}(q))$ and $\mathfrak{X}(GL_{2m}(q), GL_{2m}(q)/GL_m(q^2))$, which is considered as a possible candidate of higher dimensional analogue of the finite upper half plane. Though I have not found such a connection yet, recently I determined the exact decomposition of the permutation character $1_{GL_2(q^2)}^{GL_4(q)}$ into the irreducible characters. One obtains the list in the following tables:

The Decomposition of $1_{GL_2(q^2)}^{GL_4(q)}$, with q : odd.

Type	Degree	Frequency
$\mathrm{I}^{(1^4)}$	1	1
$\mathrm{I}^{(2^2)}$	$q^2(q^2+1)$	2
$\mathrm{I}^{(4)}$	q^6	1
$\mathrm{I}^{(1^2)}\mathrm{I}^{(1^2)}$	$(q^2+1)(q^2+q+1)$	$\frac{q-3}{2}$
$\mathrm{I}^{(2)}\mathrm{I}^{(1^2)}$	$q(q^2+1)(q^2+q+1)$	1
$\mathrm{I}^{(2)}\mathrm{I}^{(2)}$	$q^2(q^2+1)(q^2+q+1)$	$\frac{q-3}{2}$
$\mathrm{I}^{(1^2)}\mathrm{I}^{(1)}\mathrm{I}^{(1)}$	$(q+1)(q^2+1)(q^2+q+1)$	$\frac{q-3}{2}$
$\mathrm{I}^{(2)}\mathrm{I}^{(1)}\mathrm{I}^{(1)}$	$q(q+1)(q^2+1)(q^2+q+1)$	$\frac{q-3}{2}$
$\mathrm{I}^{(1)}\mathrm{I}^{(1)}\mathrm{I}^{(1)}\mathrm{I}^{(1)}$	$(q+1)^2(q^2+1)(q^2+q+1)$	$\frac{(q-3)(q-5)}{8}$
$\mathrm{I}^{(1^2)}\mathrm{II}^{(1)}$	$(q-1)(q^2+1)(q^2+q+1)$	$\frac{q-1}{2}$
$\mathrm{I}^{(2)}\mathrm{II}^{(1)}$	$q(q-1)(q^2+1)(q^2+q+1)$	$\frac{q-1}{2}$
$\mathrm{I}^{(1)}\mathrm{I}^{(1)}\mathrm{II}^{(1)}$	$(q-1)(q+1)(q^2+1)(q^2+q+1)$	$\frac{(q-1)(q-3)}{4}$
$\mathrm{II}^{(1^2)}$	$(q-1)^2(q^2+q+1)$	$\frac{q-1}{2}$
$\mathrm{II}^{(2)}$	$q^2(q-1)^2(q^2+q+1)$	$\frac{q-1}{2}$
$\mathrm{II}^{(1)}\mathrm{II}^{(1)}$	$(q-1)^2(q^2+1)(q^2+q+1)$	$\frac{(q-1)(q-3)}{8} + \frac{(q-1)^2}{4}$
$\mathrm{IV}^{(1)}$	$(q-1)^3(q+1)(q^2+q+1)$	$\frac{(q-1)(q+1)}{4}$

\# of irreducible characters $= q(q+1)$

The Decomposition of $1_{GL_2(q^2)}^{GL_4(q)}$, *with* q : *even.*

Type	Degree	Frequency
$\mathrm{I}^{(1^4)}$	1	1
$\mathrm{I}^{(2^2)}$	$q^2(q^2+1)$	1
$\mathrm{I}^{(1^2)}\mathrm{I}^{(1^2)}$	$(q^2+1)(q^2+q+1)$	$\frac{q-2}{2}$
$\mathrm{I}^{(2)}\mathrm{I}^{(2)}$	$q^2(q^2+1)(q^2+q+1)$	$\frac{q-2}{2}$
$\mathrm{I}^{(1^2)}\mathrm{I}^{(1)}\mathrm{I}^{(1)}$	$(q+1)(q^2+1)(q^2+q+1)$	$\frac{q-2}{2}$
$\mathrm{I}^{(1)}\mathrm{I}^{(1)}\mathrm{I}^{(1)}\mathrm{I}^{(1)}$	$(q+1)^2(q^2+1)(q^2+q+1)$	$\frac{(q-2)(q-4)}{8}$
$\mathrm{I}^{(1^2)}\mathrm{II}^{(1)}$	$(q-1)(q^2+1)(q^2+q+1)$	$\frac{q}{2}$
$\mathrm{I}^{(1)}\mathrm{I}^{(1)}\mathrm{II}^{(1)}$	$(q-1)(q+1)(q^2+1)(q^2+q+1)$	$\frac{q(q-2)}{4}$
$\mathrm{II}^{(1^2)}$	$(q-1)^2(q^2+q+1)$	$\frac{q}{2}$
$\mathrm{II}^{(2)}$	$q^2(q-1)^2(q^2+q+1)$	$\frac{q}{2}$
$\mathrm{II}^{(1)}\mathrm{II}^{(1)}$	$(q-1)^2(q^2+1)(q^2+q+1)$	$\frac{q(q-2)}{8}+\frac{q(q-2)}{4}$
$\mathrm{IV}^{(1)}$	$(q-1)^3(q+1)(q^2+q+1)$	$\frac{q^2}{4}$

\# of irreducible characters $= q(q+1)$

In these tables, types of irreducible characters are described in terms of pairs of monic irreducible polyomials over $\mathbb{F}_q$ and partitions (cf. Macdonald [11, Chapter IV.]). It follows that the association scheme $\mathfrak{X}(GL_4(q), GL_4(q)/GL_2(q^2))$ is a (commutative) association scheme of class $q(q+1)$.

Remark 2. The association scheme $\mathfrak{X}(GO_3(q), \Omega_3(q))$ (for even q) is isomorphic to the association scheme $\mathfrak{X}(PGL_2(q), PGL_2(q)/D_{2(q-1)})$, where $D_{2(q-1)}$ is the dihedral group of order $2(q-1)$. This association scheme is obtained by the action of the projective general linear group $PGL_2(q)$ on the set of two-element subsets of the projective geometry $PG(1,q)$, and is studied by de Caen - van Dam [7]. According to [7], the association scheme $\mathfrak{X}(PGL_2(q), PGL_2(q)/D_{2(q-1)})$ has the following subschemes:

- subschemes defined by the action of the overgroup $P\Gamma L_2(q)$,
- for $q = 4^f$ ($f \geq 2$), a subscheme of class 4 whose character table P is given as follows:

$$P = \begin{bmatrix} 1 & 2(4^f-1) & (2^{f-1}-1)(4^f-1) & 2^{f-1}(4^f-1) & 2^f(2^{f-1}-1)(4^f-1) \\ 1 & 4^f-3 & 2-2^f & -2^f & -2^f(2^f-2) \\ 1 & -2 & 2^{f-1}(2^f-1)+1 & -2^{f-1}(2^f+1) & 2^f \\ 1 & -2 & (2^{f-1}-1)(2^f-1) & 2^{f-1}(2^f-1) & -2^f(2^f-2) \\ 1 & -2 & 1-2^f & 0 & 2^f \end{bmatrix},$$

where $P\Gamma L_2(q)$ is the semidirect product of $PGL_2(q)$ with the Galois group $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$. More precisely, the existence of the above 4-class subscheme was a conjecture, and this cojecture was proved in [14]. It follows from Theorem 4.1.1 and Theorem 4.2.1 that we have found another kind of subschemes of $\mathfrak{X}(PGL_2(q), PGL_2(q)/D_{2(q-1)})$.

Remark 3. Professor E. Bannai has pointed out that some graphs obtained from the relations of our association schemes are *Ramanujan graphs*, that is, regular graphs having good *expansion constants* (cf. Terras [15, Chapter 3.]).

References

[1] E. Bannai, Subschemes of Some Association Schemes, *J. Algebra* **144** (1991),167-188.

[2] E. Bannai, S. Hao, and S.-Y. Song, Character Tables of the Association Schemes of Finite Orthogonal Groups Acting on the Nonisotropic Points, *J. Combin. Theory Ser.* **A 54** (1990),164-200.

[3] E. Bannai, S. Hao, S.-Y. Song, and H. Wei, Character Tables of Certain Association Schemes Coming From Finite Unitary and Symplectic Groups, *J. Algebra* **144** (1991),189-213.

[4] E. Bannai and T. Ito, "*Algebraic Combinatorics* **I**," Benjamin/Cummings, Menlo Park, CA, 1984.

[5] E. Bannai, N. Kawanaka, and S.-Y. Song, The Character Table of the Hecke Algebra $\mathscr{H}(GL_{2n}(\mathbb{F}_q), Sp_{2n}(\mathbb{F}_q))$, *J. Algebra* **129** (1990),320-366.

[6] E. Bannai, W.M. Kwok, and S.-Y. Song, Ennola Type Dualities in the Character Tables of Some Association Schemes, *Mem. Fac. Sci. Kyushu Univ. Ser.* **A 44** (1990),129-143.

[7] D. de Caen and E. R. van Dam, Fissioned triangular schemes via the cross-ratio, to appear in *Europ. J. Combin.*

[8] J.H. Conway *et al.*, "**ATLAS** *of Finite Groups*," Clarendon Press, Oxford, 1985.

[9] W.M. Kwok, Character Tables of Association Schemes of Affine Type, *Europ. J. Combin.* **13** (1992),167-185.

[10] R. Lidl and H. Niederreiter, "*Finite Fields*" 2nd ed., Encyclopedia of Mathematics and its Applications **20**, Cambridge University Press, 1997.

[11] I.G. Macdonald, "*Symmetric Functions and Hall Polynomials*" 2nd ed., Oxford Mathmatical Monographs, Oxford University Press, 1995

[12] A. Munemasa, "*The Geometry of Orthogonal Groups over Finite Fields*," JSPS-DOST Lecture Notes in Math. **3**, Sophia University, Tokyo, 1996.

[13] G. Seroussi and A. Lempel, Factorization of Symmetric Matrices and Trace-orthogonal Bases in Finite Fields, *SIAM J. Comput.* **9** (1980),758-767.

[14] H. Tanaka, A 4-class Subscheme of the Association Scheme Coming from the Action of $PGL(2, 4^f)$, to appear in *Europ. J. Combin.*

[15] A. Terras, "*Fourier Analysis on Finite Groups and Applications*," London Math. Soc. Student Texts **43**, Cambridge University Press, 1999.