Vertex subsets with minimal width and dual width in Q-polynomial distance-regular graphs

Hajime Tanaka

Tohoku University

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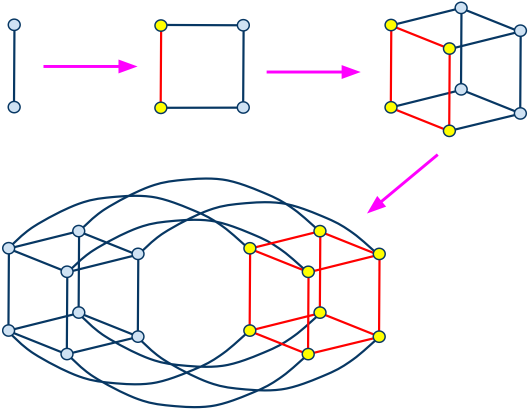
Every face of a hypercube is a hypercube...



Study this situation and generalize it to Q-polynomial distance-regular graphs.

Q-polynomial distance-regular graphs

- $\Gamma = (X, R)$: a connected simple graph with diameter d and valency k
- ∂ : the path-length distance function
- Define $A_0, A_1, \ldots, A_d \in \mathbb{R}^{X \times X}$ by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{otherwise} \end{cases}$$

- Γ is distance-regular if there are integers a_i, b_i, c_i such that

$$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \ (0 \leqslant i \leqslant d)$$

where $A_{-1} = A_{d+1} = 0$.

Q-polynomial distance-regular graphs

- $\Gamma = (X, R)$: a distance-regular graph with diameter d
- We set $A := A_1$ (the adjacency matrix of Γ).
- $\theta_0 := k, \theta_1, \ldots, \theta_d$: the distinct eigenvalues of A
- E_i : the orthogonal projection onto the eigenspace of A with eigenvalue θ_i
- $\mathbb{R}[A] = \langle A_0, \ldots, A_d \rangle = \langle E_0, \ldots, E_d \rangle$: the Bose–Mesner algebra of Γ

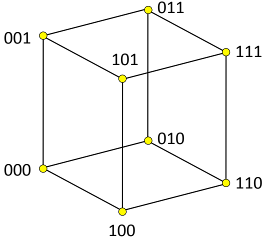
- Γ is Q-polynomial with respect to $\{E_i\}_{i=0}^d$ if there are scalars a_i^*, b_i^*, c_i^* $(0 \leqslant i \leqslant d)$ such that $b_{i-1}^* c_i^* \neq 0$ $(1 \leqslant i \leqslant d)$ and

$$|X| E_1 \circ E_i = b_{i-1}^* E_{i-1} + a_i^* E_i + c_{i+1}^* E_{i+1} \ (0 \leqslant i \leqslant d)$$

where $E_{-1} = E_{d+1} = 0$ and $\circ$ is the Hadamard product.

Example: hypercubes

- $X = \{0, 1\}^d$
- $x \sim_R y \iff |\{i : x_i \neq y_i\}| = 1$
- $\Gamma = Q_d = (X, R)$: the hypercube
- Q_3 :



- $\mathcal{P}$: the set of faces of Q_d
- $u \preccurlyeq v \iff u \supseteq v \quad (u, v \in \mathcal{P})$
- $H(d, 2) = (\mathcal{P}, \preccurlyeq)$: the binary Hamming matroid
- $X = \{0, 1\}^d = \mathrm{top}(\mathcal{P})$: the top fiber of $H(d, 2)$

Five classical families of Q-polynomial DRGs

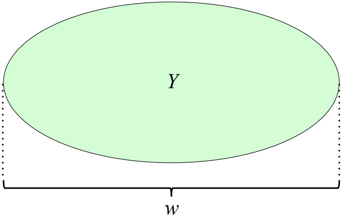
... are associated with nice semilattice structures:

$(\mathcal{P}, \preccurlyeq)$	$\mathrm{top}(\mathcal{P})$
truncated Boolean algebra	Johnson graph
Hamming matroid	Hamming graph
truncated projective geometry	Grassmann graph
attenuated space	bilinear forms graph
classical polar space	dual polar graph

Width and dual width (Brouwer et al., 2003)

- $\Gamma = (X, R)$: a distance-regular graph with diameter d
- $A_0, A_1, \ldots, A_d$: the distance matrices
- $E_0, E_1, \ldots, E_d$: the primitive idempotents of $\mathbb{R}[A]$
- Suppose Γ is Q-polynomial with respect to $\{E_i\}_{i=0}^{d}$.

- $Y \subseteq X$: a nonempty subset of X
- $\chi \in \mathbb{R}^X$: the characteristic vector of Y
- $w = \max\{i : \chi^{\mathsf{T}} A_i \chi \neq 0\}$: the width of Y
- $w^* = \max\{i : \chi^{\mathsf{T}} E_i \chi \neq 0\}$: the dual width of Y



Theorem (Brouwer–Godsil–Koolen–Martin, 2003)

We have $w + w^* \geqslant d$. If equality holds then Y is completely regular, and the induced subgraph Γ_Y on Y is a Q-polynomial distance-regular graph with diameter w provided it is connected.

Definition

We call Y a descendent of Γ if $w + w^* = d$.

Theorem (T.)

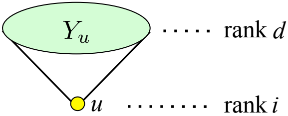
Let Y be a descendent of Γ and suppose Γ_Y is connected. Then a nonempty subset of Y is a descendent of Γ_Y if and only if it is a descendent of Γ.

Examples of descendents, i.e., $w + w^* = d$

- $w = 0$: $Y = \{x\} \quad (x \in X)$
- $w = d$: $Y = X$
- $w = 1$: Delsarte cliques ($\Rightarrow \theta_d = \theta_{\min}$)

Examples of descendents, i.e., $w + w^* = d$

- Γ : a Johnson, Hamming, Grassmann, bilinear forms, or a dual polar graph
- $(\mathcal{P}, \preccurlyeq)$: the associated semilattice
- $u \in \mathcal{P}$: rank i
- $Y_u := \{x \in X : u \preccurlyeq x\}$



Theorem (Brouwer et al., 2003; T., 2006)

Every descendent of Γ is isomorphic to some Y_u under the full automorphism group of Γ.

Classical parameters

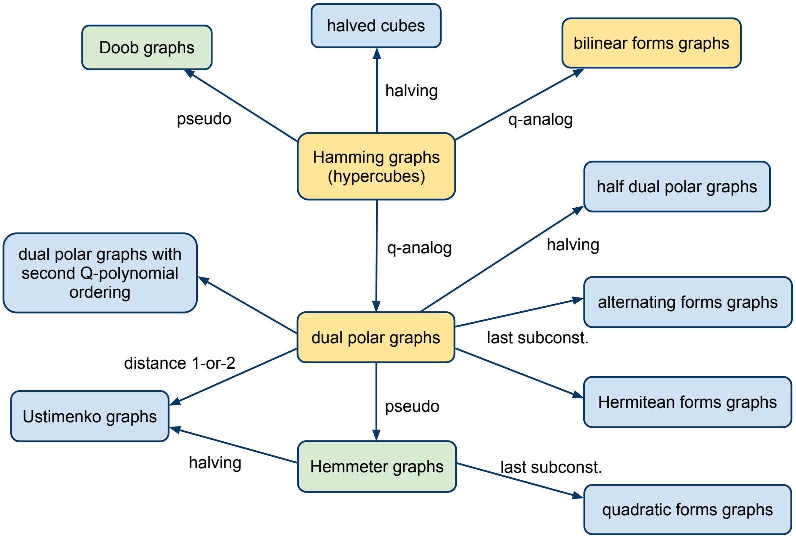
- We say Γ has classical parameters (d, q, α, β) if

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix}_q - \begin{bmatrix} i \\ 1 \end{bmatrix}_q \right) \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_q \right), \quad c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_q \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_q \right)$$

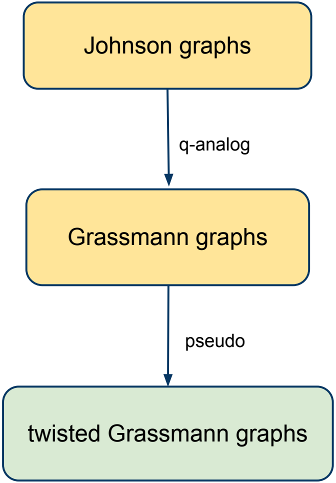
for $0 \leqslant i \leqslant d$, where $\begin{bmatrix} i \\ j \end{bmatrix}_q$ is the q-binomial coefficient.

Currently, there are 15 known infinite families of distance-regular graphs with classical parameters and with unbounded diameter.

The families related to Hamming graphs



The families related to Johnson graphs



- $Y \subseteq X$: a descendent of Γ, i.e., $w + w^* = d$
- Γ_Y : the induced subgraph on Y

Theorem (T.)

Suppose $1 < w < d$. Then Y is convex (i.e., geodetically closed) precisely when Γ has classical parameters.

Theorem (T.)

If Γ has classical parameters (d, q, α, β) then Γ_Y has classical parameters (w, q, α, β). The converse also holds, provided $w \geqslant 3$.

A characterization of the five classical families

- Γ : a Q-polynomial distance-regular graph with $d \geqslant 3$
- $\mathcal{P}$: a set of descendents of Γ
- We say $\mathcal{P}$ satisfies $(\mathrm{UD})_i$ if any two $x, y \in X$ with $\partial(x, y) = i$ are contained in a unique $Y \in \mathcal{P}$ with width i.

Theorem (T.)

Suppose the following hold:

1. Γ *has classical parameters.*
2. $\mathcal{P}$ *satisfies* $(\mathrm{UD})_i$ *for* $0 \leqslant i \leqslant d$.
3. $Y_1 \cap Y_2 \in \mathcal{P}$ *for all* $Y_1, Y_2 \in \mathcal{P}$ *such that* $Y_1 \cap Y_2 \neq \emptyset$.

Then $\mathcal{P}$*, together with the partial order defined by reverse inclusion, forms a regular quantum matroid in the sense of Terwilliger.*

A characterization of the five classical families

$(\mathcal{P}, \preccurlyeq)$	$\mathrm{top}(\mathcal{P})$
truncated Boolean algebra	Johnson graph
Hamming matroid	Hamming graph
truncated projective geometry	Grassmann graph
attenuated space	bilinear forms graph
classical polar space	dual polar graph

Theorem (Terwilliger, 1996)

A regular quantum matroid of rank at least four is isomorphic to one of the above five examples.

A characterization of the five classical families

Corollary (T.)

Suppose the following hold:

1. Γ *has classical parameters.*
2. $\mathcal{P}$ *satisfies* $(\mathrm{UD})_i$ *for* $0 \leqslant i \leqslant d$.
3. $Y_1 \cap Y_2 \in \mathcal{P}$ *for all* $Y_1, Y_2 \in \mathcal{P}$ *such that* $Y_1 \cap Y_2 \neq \emptyset$.

If $d \geqslant 4$ *then* Γ *is either a Johnson, Hamming, Grassmann, bilinear forms or dual polar graph.*

Remark

If $\mathcal{P}$ is the set of descendents of Γ, then 1, 2 imply 3.

Conjecture

If $\mathcal{P}$ is the set of descendents of Γ, then 2 imply 1.

Classification of descendents

$\cdots$ is now complete for all 15 families (T.).

- $\mathcal{P}$: the set of descendents of Γ
- $w(\mathcal{P}) = \{w(Y) : Y \in \mathcal{P}\}$

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Γ	$w(\mathcal{P})\backslash\{0,d\}$
Johnson	$\{1,2,\ldots,d-1\}$
Hamming	$\{1,2,\ldots,d-1\}$
Grassmann	$\{1,2,\ldots,d-1\}$
bilinear forms	$\{1,2,\ldots,d-1\}$
dual polar	$\{1,2,\ldots,d-1\}$
Doob	$\{1,2,\ldots,d-1\}$
Hemmeter	$\{1,2,\ldots,d-1\}$
twisted Grassmann	$\{1,2,\ldots,d-1\}$
halved cube	$\{1,d-1\}$ or $\emptyset$
Hermitean forms	$\emptyset$
alternating forms	$\{1,d-1\}$ or $\emptyset$
quadratic forms	$\{1,d-1\}$ or $\emptyset$
dual polar with 2nd Q-poly.	$\emptyset$
half dual polar	$\{1,d-1\}$ or $\emptyset$
Ustimenko	$\{1,d-1\}$ or $\emptyset$