# The Terwilliger algebra of a $Q$-polynomial distance-regular graph with respect to a set of vertices

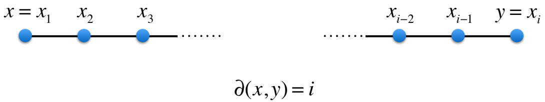
Hajime Tanaka
(joint work with Rie Tanaka and Yuta Watanabe)

**R**esearch **C**enter for **P**ure and **A**pplied **M**athematics
**G**raduate **S**chool of **I**nformation **S**ciences
Tohoku University

November 29, 2014
Algebraic Combinatorics Workshop

# Notation

- $\Gamma = (X, R)$ : a finite connected simple graph
  - $X$ : the vertex set
  - $R$ : the edge set ($=$ a set of $2$-element subsets of $X$)
- $\partial$ : the path-length distance on $X$



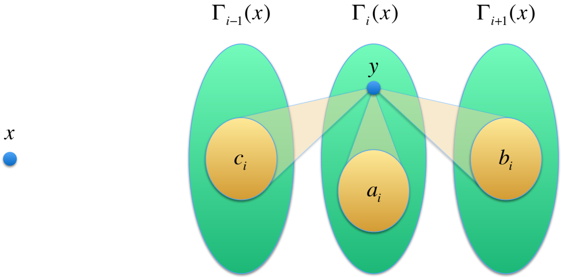
- $D := \max\{\partial(x, y) : x, y \in X\}$ : the diameter of $\Gamma$
- $\Gamma_i(x) := \{y \in X : \partial(x, y) = i\}$ : the $i^{\text{th}}$ subconstituent

# Distance-regular graphs

- $\Gamma$ : distance-regular

  $\overset{\text{def}}{\Longleftrightarrow} \exists a_i, b_i, c_i \ (0 \leqslant i \leqslant D)$ s.t. $\forall x, y \in X$ :
  - $|\Gamma_{i-1}(x) \cap \Gamma_1(y)| = c_i$
  - $|\Gamma_i(x) \cap \Gamma_1(y)| = a_i$
  - $|\Gamma_{i+1}(x) \cap \Gamma_1(y)| = b_i$

  where $\partial(x, y) = i$.



# The adjacency algebra

- $\mathrm{Mat}_X(\mathbb{C})$ : the set of square matrices over $\mathbb{C}$ index by $X$
- The $i^{\text{th}}$ distance matrix $A_i \in \mathrm{Mat}_X(\mathbb{C})$ is

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } \partial(x,y) = i \\ 0 & \text{otherwise} \end{cases}$$

[ Note: $A_0 = I$ ]

- $A_0, A_1, \dots, A_D$ satisfy the three-term recurrence

$$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \quad (0 \leqslant i \leqslant D)$$

where $A_{-1} = A_{D+1} = 0$.

# The adjacency algebra

- Recall the three-term recurrence

  $$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \quad (0 \leqslant i \leqslant D)$$

  where $A_{-1} = A_{D+1} = 0$.
- $\boldsymbol{M} := \mathbb{C}[A_1] \subseteq \mathrm{Mat}_X(\mathbb{C})$ : the adjacency algebra of $\Gamma$
  - $\exists v_i \in \mathbb{Q}[t]$ s.t. $\deg v_i = i$ and $A_i = v_i(A_1) \quad (0 \leqslant i \leqslant D)$
  - $\boldsymbol{M} = \langle A_0, A_1, \ldots, A_D \rangle$
  - $A_1$ has $D+1$ distinct eigenvalues $\theta_0, \theta_1, \ldots, \theta_D \in \mathbb{R}$.

# The $Q$-polynomial property

- Recall
  - $\theta_0, \theta_1, \ldots, \theta_D \in \mathbb{R}$ : the distinct eigenvalues of $A_1$
  - $\Gamma$ : regular with valency $k_1 := |\Gamma_1(x)| \, (= b_0)$
- Always set $\theta_0 = k_1$.
- $E_\ell \in \mathrm{Mat}_X(\mathbb{C})$ : the orthogonal projection onto the eigenspace of $\theta_\ell$ [ Note: $E_0 = \frac{1}{|X|} J$ ($J$ : the all-ones matrix) ]
- $\boldsymbol{M} = \mathbb{C}[A_1] = \langle A_0, A_1, \ldots, A_D \rangle = \langle E_0, E_1, \ldots, E_D \rangle$
- $E_0, E_1, \ldots, E_D$ : the primitive idempotents of $\boldsymbol{M}$

# The $Q$-polynomial property

- Recall the three-term recurrence

$$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \quad (0 \leqslant i \leqslant D).$$

- $\Gamma$ : $Q$-polynomial w.r.t. $\{E_\ell\}_{\ell=0}^{D}$

$\overset{\text{def}}{\Longleftrightarrow} \exists a_\ell^*, b_\ell^*, c_\ell^*$ $(0 \leqslant \ell \leqslant D)$ s.t. $b_{\ell-1}^* c_\ell^* \neq 0 \ (1 \leqslant \ell \leqslant D)$ and

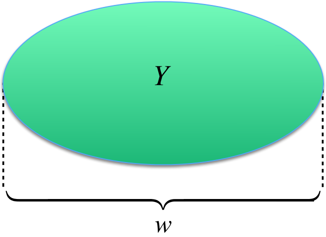
$$|X| \, E_1 \circ E_\ell = b_{\ell-1}^* E_{\ell-1} + a_\ell^* E_\ell + c_{\ell+1}^* E_{\ell+1} \quad (0 \leqslant \ell \leqslant D)$$

where $E_{-1} = E_{D+1} = 0$ and $\circ$ is the entrywise product.

# Width and dual width (Brouwer et al., 2003)

- We shall assume $\Gamma$ is a $Q$-polynomial DRG.

- $Y \subseteq X$ : a nonempty subset of $X$
- $\chi \in \mathbb{C}^X$ : the characteristic vector of $Y$
- $w = \max\{i : \chi^{\mathsf{T}} A_i \chi \neq 0\}$ : the width of $Y$
- $w^* = \max\{\ell : \chi^{\mathsf{T}} E_\ell \chi \neq 0\}$ : the dual width of $Y$



# Width and dual width (Brouwer et al., 2003)

**Theorem (Brouwer–Godsil–Koolen–Martin, 2003)**

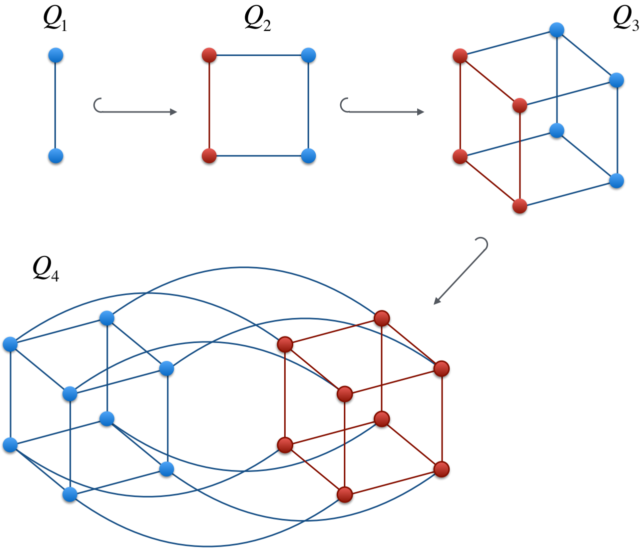
- *We have*
$$w + w^* \geqslant D.$$
- *If $w + w^* = D$ then $Y$ is completely regular, and the induced subgraph $\Gamma_Y$ on $Y$ is a $Q$-polynomial DRG with diameter $w$ provided it is connected.*

- $Y$ : a descendent of $\Gamma \overset{\text{def}}{\iff} w + w^* = D$
- Descendents play a role in the Assmus–Mattson theorem (T., 2009) and also in the Erdős–Ko–Rado theorem (T., 2006, 2012).

# Some descendents

- $w = 0$ : $Y = \{x\} \ \ (x \in X)$
- $w = D$ : $Y = X$
- $w = 1$ : Delsarte cliques ($\Longrightarrow \theta_D = \theta_{\min}$) i.e., $|Y| = 1 - \frac{k_1}{\theta_D}$

# A chain of descendents



# A poset

## Theorem (T., 2011)

- *Let $Y$ be a descendent of $\Gamma$ and suppose $\Gamma_Y$ is connected. Then a nonempty subset of $Y$ is a descendent of $\Gamma_Y$ if and only if it is a descendent of $\Gamma$.*

- $\mathscr{L}$ : the set of isomorphism classes of $Q$-polynomial DRGs
- $[\Delta] \preccurlyeq [\Gamma] \overset{\text{def}}{\iff} \exists Y$ : a descendent of $\Gamma$ s.t. $[\Delta] = [\Gamma_Y]$
- $(\mathscr{L}, \preccurlyeq)$ : a poset

# The structure of $(\mathscr{L}, \preccurlyeq)$

- The classification of descendents is complete for the $15$ known infinite families of DRGs with unbounded diameter and with classical parameters (BGKM, 2003; T., 2006, 2011).
- The ideal $\mathscr{I}_{[\Gamma]} = \{[\Delta] \in \mathscr{L} : [\Delta] \preccurlyeq [\Gamma]\}$ is known if $\Gamma$ belongs to one of the above families.

# The structure of $(\mathscr{L}, \preccurlyeq)$

**Problem**

- Determine the filter $\mathscr{F}_{[\Gamma]} = \{[\Delta] \in \mathscr{L} : [\Gamma] \preccurlyeq [\Delta]\}$

- This has been solved at the parameteric level.
- The generic case is described in terms of 5 scalars (besides $D$) $q, r_1, r_2, s, s^*$ where $r_1 r_2 = s s^* q^{D+1}$ (Leonard, 1982).

**Theorem (T., 2009, 2011)**

- *Suppose $[\Gamma] \preccurlyeq [\Delta]$ and $\Delta$ has diameter $C \geqslant D$. If $D \geqslant 3$ then the scalars corresponding to $\Delta$ are*

$$q, r_1, r_2, sq^{D-C}, s^*.$$

$r_1 r_2 = \left(sq^{D-C}\right) s^* q^{C+1}$

# When $\Gamma_Y$ is connected

**Theorem (Brouwer–Godsil–Koolen–Martin, 2003)**

- *We have*
$$w + w^* \geqslant D.$$
- *If $w + w^* = D$ then $Y$ is completely regular, and the induced subgraph $\Gamma_Y$ on $Y$ is a $Q$-polynomial DRG with diameter $w$ provided it is connected.*

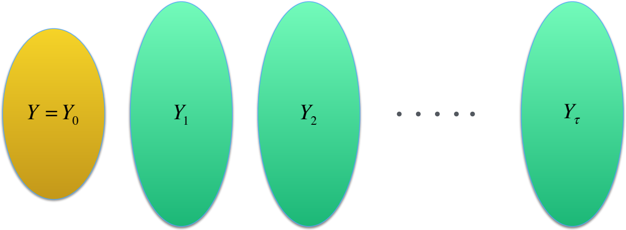
**Theorem (T., 2011)**

- *Let $Y$ be a descendent of $\Gamma$. Then $\Gamma_Y$ is connected if and only if $q \neq -1$, or $q = -1$ and $w^*$ is even.*

▸ Jump to 21

# The Terwilliger algebra

- $Y \subseteq X$ : a nonempty subset of $X$
- $Y_i = \{z \in X : \partial(z, Y) = i\}$
- $\tau = \max\{i : Y_i \neq \emptyset\}$ : the covering radius of $Y$



the distance partition of $X$

# The Terwilliger algebra

$i^{\mathrm{th}}$ “dual idempotent”

- $\chi_i \in \mathbb{C}^X$ : the characteristic vector of $Y_i$ $(0 \leqslant i \leqslant \tau)$
- $E_i^* = \mathrm{Diag}(\chi_i) \in \mathrm{Mat}_X(\mathbb{C})$ $(0 \leqslant i \leqslant \tau)$,

$$(E_i^*)_{zz} = \begin{cases} 1 & \text{if } z \in Y_i, \\ 0 & \text{otherwise,} \end{cases} \qquad (z \in X).$$

- $\boldsymbol{T} = \boldsymbol{T}(Y) = \mathbb{C}[A_1, E_0^*, \ldots, E_\tau^*]$ : the Terwilliger algebra with respect to $Y$ (Martin–Taylor, 1997; Suzuki, 2005)
- $Y = \{x\} \implies \boldsymbol{T} = \boldsymbol{T}(x)$ : the Terwilliger algebra with respect to $x$ (Terwilliger, 1992)

“subconstituent algebra”

# The case when $Y$ is a descendent

- We shall assume $Y$ is a descendent of $\Gamma$.

“dual degree”

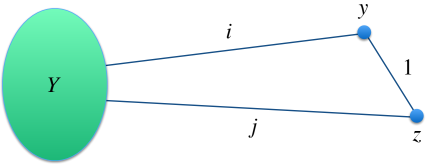
- We have $\tau = |\{\ell \neq 0 : \chi^{\mathsf{T}} E_\ell \chi \neq 0\}| = w^*$.

Delsarte (1973) BGKM (2003)

- $\boldsymbol{T} = \mathbb{C}[A_1, E_0^*, \ldots, E_{w^*}^*]$

# The dual adjacency matrix

- $E_i^* A_1 E_j^* = 0$ if $|i-j| > 1$



- $A_1^* = \frac{|X|}{|C|} \operatorname{Diag}(E_1 \chi) \in \operatorname{Mat}_X(\mathbb{C})$ : the dual adjacency matrix
- $Y$ : completely regular $\Longrightarrow A_1^* \in \boldsymbol{M}^* := \langle E_0^*, E_1^*, \ldots, E_{w^*}^* \rangle$

"dual Bose–Mesner algebra"

**Lemma (cf. Cameron–Goethals–Seidel, 1978)**

- $E_i A_1^* E_j = 0$ *if* $|i-j| > 1$

# Tridiagonal pairs

- $W$ : a finite-dimensional complex vector space
- $\mathfrak{a}, \mathfrak{a}^* \in \mathrm{End}(W)$ : diagonalizable
- $(\mathfrak{a}, \mathfrak{a}^*)$ : a tridiagonal pair (Ito–Tanabe–Terwilliger, 2001)

$\overset{\text{def}}{\iff}$

  - $\exists W_0, W_1, \ldots, W_d$ : an ordering of the eigenspaces of $A$ s.t.

$$\mathfrak{a}^* W_i \subset W_{i-1} + W_i + W_{i-1} \quad (0 \leqslant i \leqslant d);$$

  - $\exists W_0^*, W_1^*, \ldots, W_{d^*}^*$ : an ordering of the eigenspaces of $A^*$ s.t.

$$\mathfrak{a} W_i^* \subset W_{i-1}^* + W_i^* + W_{i-1}^* \quad (0 \leqslant i \leqslant d^*);$$

  - $W$ : irreducible as a $\mathbb{C}[\mathfrak{a}, \mathfrak{a}^*]$-module.

## Proposition (Ito–Tanabe–Terwilliger, 2001)

- $d = d^*$.

# Do irreducible $T$-modules afford tridiagonal pairs?

- $E_i^* A_1 E_j^* = 0$ if $|i-j| > 1$
- $E_i A_1^* E_j = 0$ if $|i-j| > 1$

- $W$ : an irreducible $\boldsymbol{T}$-module
- $A_1 E_i^* W \subset E_{i-1}^* W + E_i^* W + E_{i+1}^* W$
- $A_1^* E_i W \subset E_{i-1} W + E_i W + E_{i+1} W$
- If $\boldsymbol{M}^* = \mathbb{C}[A_1^*]$ then $W$ is irreducible as a $\mathbb{C}[A_1, A_1^*]$-module.

$\boldsymbol{T} = \mathbb{C}[A_1, A_1^*]$

**Theorem**

- *Every irreducible $\boldsymbol{T}$-module affords a tridiagonal pair if and only if $q \neq -1$, or $q = -1$ and $w$ is even.*

Jump to 15

# Some general results

- We shall assume $q \neq -1$.

- $W$ : an irreducible $\boldsymbol{T}$-module

- $\rho = \min\{i : E_i^* W \neq 0\}$ : the endpoint of $W$
- $\rho^* = \min\{\ell : E_\ell W \neq 0\}$ : the dual endpoint of $W$
- $d = |\{i : E_i^* W \neq 0\}| = |\{\ell : E_\ell W \neq 0\}|$ : the diameter of $W$

- $\{i : E_i^* W \neq 0\} = \{\rho, \ldots, \rho + d\} \subset \{0, 1, \ldots, w^*\}$
- $\{\ell : E_\ell W \neq 0\} = \{\rho^*, \ldots, \rho^* + d\} \subset \{0, 1, \ldots, D\}$

**Proposition (cf. Caughman, 1999)**

- $2\rho + d \geqslant w^*$
- $2\rho^* + d \geqslant w^*$

# Some general results

- $\rho + d \leqslant w^*$
- $\rho^* + d \leqslant D$
- $2\rho + d \geqslant w^*$
- $2\rho^* + d \geqslant w^*$

- $\eta := \underset{\substack{\rotatebox{90}{$\geqslant$} \\ w^*}}{\rho} + \underset{\substack{\rotatebox{90}{$\geqslant$} \\ D}}{\underline{\rho^* + d}} - w^*$ : the displacement of $W$

- $0 \leqslant \eta \leqslant D$

- We may generalize the displacement and split decompositions of $\mathbb{C}^X$ due to Terwilliger (2005).
- In particular, it is likely that $U_q(\widehat{\mathfrak{sl}}_2) \twoheadrightarrow \boxtimes_q \overset{\exists}{\to} \boldsymbol{T}$ when $\Gamma$ is a forms graph (cf. Ito–Terwilliger, 2009).

“$q$-tetrahedron algebra”

# Some general results

- $W$ : thin $\overset{\text{def}}{\iff} \dim E_i^* W \leqslant 1 \ \ (0 \leqslant i \leqslant D)$
  $\iff$ the associated tridiagonal pair is a Leonard pair

### Theorem (Hosoya–Suzuki, 2007)

- *There are precisely $w+1$ inequivalent irreducible $\boldsymbol{T}$-modules in $\mathbb{C}^X$ with $\rho = 0$.*
- *Each of such modules is thin and is generated by an eigenvector of $\Gamma_Y$ in $\mathbb{C}^Y = E_0^* \mathbb{C}^X$.*

a $Q$-polynomial DRG with diameter $w$

# Hamming graphs

- $[q] = \{0, 1, \ldots, q-1\}$ $(q \geqslant 2)$
- $X = [q]^D$
- $y \sim z \overset{\text{def}}{\iff} |\{i : y_i \neq z_i\}| = 1$
- $\Gamma = H(D, q)$ : the Hamming graph

- The structure of $\boldsymbol{T}(x)$ has been well studied.
- $H(D, 2) = \mathcal{Q}_D \implies U(\mathfrak{sl}_2) \overset{\exists}{\twoheadrightarrow} \boldsymbol{T}(x)$ (Go, 2002)
- $H(D, q)$ $(q \geqslant 3) \implies$ The method for the Doob graphs (Tanabe, 1997) works as well.

# Hamming graphs

- $n \in \{0, 1, \ldots, D\}$
- $Y = \{z \in X : z_1 = \cdots = z_n = 0\}$ : a descendent with $w = D - n$, $w^* = n$

$$z = (\underbrace{0, \ldots, 0}_{n} \,|\, \underbrace{*, \ldots, *}_{D-n})$$

- $\Gamma_Y \cong H(D - n, q)$

## Theorem (Brouwer–Godsil–Koolen–Martin, 2003)

- *Every descendent of $\Gamma = H(D, q)$ with $w^* = n$ is isomorphic (under $\mathrm{Aut}\,\Gamma$) to $Y$ above.*

# Hamming graphs

- $z = (\underbrace{0, 0, 0, \ldots, 0}_{n} \,|\, \underbrace{*, \ldots, *}_{D-n}) \in Y = Y_0$
- $z = (\underbrace{1, 0, 0, \ldots, 0}_{n} \,|\, \underbrace{*, \ldots, *}_{D-n}) \in Y_1$
- $z = (\underbrace{1, 1, 0, \ldots, 0}_{n} \,|\, \underbrace{*, \ldots, *}_{D-n}) \in Y_2$

- $Y_i = \Gamma'_i(\mathbf{0}) \times [q]^{D-n} \ (0 \leqslant i \leqslant n)$

  where $\Gamma' = H(n, q)$ and $\mathbf{0} = (\underbrace{0, \ldots, 0}_{n})$

# Hamming graphs

- $\Gamma' = H(n,q), \quad \Gamma'' = H(D-n,q)$
- Use $'$ (resp. $''$) to denote objects associated with $\Gamma'$ (resp. $\Gamma''$).

- $Y_i = \Gamma'_i(\mathbf{0}) \times [q]^{D-n} \ (0 \leqslant i \leqslant n)$
- $E^*_i = E^{*\prime}_i \otimes I'' \in \boldsymbol{T}'(\mathbf{0}) \otimes \boldsymbol{M}'' \ (0 \leqslant i \leqslant n)$

- $A_1 = A'_1 \otimes I'' + I' \otimes A''_1 \in \boldsymbol{T}'(\mathbf{0}) \otimes \boldsymbol{M}''$
- $\boldsymbol{T} \subset \boldsymbol{T}'(\mathbf{0}) \otimes \boldsymbol{M}''$

### Theorem

- *Every irreducible* $(\boldsymbol{T}'(\mathbf{0}) \otimes \boldsymbol{M}'')$*-module is a thin irreducible* $\boldsymbol{T}$*-module.*

# Johnson graphs

- Use $\widetilde{\phantom{x}}$ to denote objects associated with $\mathcal{Q}_v = H(v,2) \ \ (v \geqslant 2D)$.
- $X = \underset{\nwarrow}{\widetilde{\Gamma}_D}(\mathbf{0}) = \{z \in [2]^v : \partial(\mathbf{0}, z) = D\}$ where $\mathbf{0} = (0, \ldots, 0)$

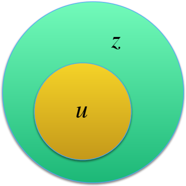
  in bijection with $\binom{[v]}{D}$

- $y \sim z \overset{\text{def}}{\iff} \partial(y,z) = 2$
- $\Gamma = J(v, D)$ : the Johnson graph

# Johnson graphs

- $n \in \{0, 1, \ldots, D\}$
- $u \in \widetilde{\Gamma}_n(\mathbf{0}) \longleftrightarrow \binom{[v]}{n}$
- $Y = \{z \in X : \partial(u, z) = D - n\}$ : a descendent with $w = D - n$, $w^* = n$

$$\begin{aligned} u &= (\overbrace{1, \ldots, 1}^{n} \,|\, \overbrace{0, \ldots, 0, 0, \ldots, 0}^{v-n}) \\ z &= (1, \ldots, 1 \,|\, \underbrace{1, \ldots, 1}_{D-n}, 0, \ldots, 0) \end{aligned}$$



- $\Gamma_Y \cong J(v - n, D - n)$

**Theorem (Brouwer–Godsil–Koolen–Martin, 2003)**

- *Every descendent of $\Gamma = J(v, D)$ with $w^* = n$ is isomorphic (under $\operatorname{Aut}\Gamma$) to $Y$ above.*

# Johnson graphs

- $u = (\underbrace{1, \ldots, 1, 1, 1}_{n} \,|\, \underbrace{0, \ldots, 0, 0, 0, 0, \ldots, 0}_{v-n})$
- $z = (\underbrace{1, \ldots, 1, 1, 1}_{n} \,|\, \underbrace{1, \ldots, 1}_{D-n}, 0, 0, 0, \ldots, 0) \;\in Y = Y_0$
- $z = (\underbrace{1, \ldots, 1, 1}_{n-1}, 0 \,|\, \underbrace{1, \ldots, 1, 1}_{D-n+1}, 0, 0, \ldots, 0) \;\in Y_1$
- $z = (\underbrace{1, \ldots, 1}_{n-2}, 0, 0 \,|\, \underbrace{1, \ldots, 1, 1, 1}_{D-n+2}, 0, \ldots, 0) \;\in Y_2$

- $Y_i = \Gamma'_{n-i}(\mathbf{0}) \times \Gamma''_{D-n+i}(\mathbf{0}) \;\; (0 \leqslant i \leqslant n)$
  where $\Gamma' = \mathcal{Q}_n$ and $\Gamma'' = \mathcal{Q}_{v-n}$

# Johnson graphs

- $\Gamma' = \mathcal{Q}_n, \quad \Gamma'' = \mathcal{Q}_{v-n}$
- Use $'$ (resp. $''$) to denote objects associated with $\Gamma'$ (resp. $\Gamma''$).

- $Y_i = \Gamma'_{n-i}(\mathbf{0}) \times \Gamma''_{D-n+i}(\mathbf{0}) \ \ (0 \leqslant i \leqslant n)$
- $E_i^* = E^{*\prime}_{n-i} \otimes E^{*\prime\prime}_{D-n+i} \in \widetilde{E}_D^* \big(\boldsymbol{T}' \otimes \boldsymbol{T}''\big) \widetilde{E}_D^*$

both w.r.t. $\mathbf{0}$

- $A_1 = \widetilde{E}_D^* \underline{\widetilde{A}_2} \widetilde{E}_D^* \ \in \widetilde{E}_D^* \big(\boldsymbol{T}' \otimes \boldsymbol{T}''\big) \widetilde{E}_D^*$

$A'_2 \otimes I'' + A'_1 \otimes A''_1 + I' \otimes A''_2$

- $\boldsymbol{T} \subset \widetilde{E}_D^* \big(\boldsymbol{T}' \otimes \boldsymbol{T}''\big) \widetilde{E}_D^*$

### Theorem

- *Every irreducible* $\big(\widetilde{E}_D^* \big(\boldsymbol{T}' \otimes \boldsymbol{T}''\big) \widetilde{E}_D^*\big)$*-module is a thin irreducible* $\boldsymbol{T}$*-module.*

# Grassmann graphs

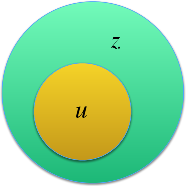
- $\mathcal{V} = \mathbb{F}_q^v \ (v \geqslant 2D)$
- $X = \begin{bmatrix} \mathcal{V} \\ D \end{bmatrix}_q \longleftarrow$ the set of $D$-dimensional subspaces of $\mathcal{V}$
- $y \sim z \overset{\text{def}}{\Longleftrightarrow} \dim(y \cap z) = D - 1$
- $\Gamma = J_q(v, D)$ : the Grassmann graph

# Grassmann graphs

- $n \in \{0, 1, \ldots, D\}$
- $u \in \begin{bmatrix} \mathcal{V} \\ n \end{bmatrix}_q$

- $Y = \{z \in X : u \leqslant z\}$ : a descendent with $w = D - n$, $w^* = n$

- $\Gamma_Y \cong J_q(v - n, D - n)$



### Theorem (T., 2006)

- *Every descendent of $\Gamma = J_q(v, D)$ with $w^* = n$ is isomorphic (under $\mathrm{Aut}\,\Gamma$) to $Y$ above.*

- $Y_i = \{z \in X : \dim(u \cap z) = n - i\}$ $(0 \leqslant i \leqslant n)$

# Grassmann graphs

- $P(\mathcal{V}) = \coprod_{i=0}^{v} \begin{bmatrix} \mathcal{V} \\ i \end{bmatrix}_q$ : the set of subspaces of $\mathcal{V}$
- $G = \mathrm{GL}(\mathcal{V}) \curvearrowright P(\mathcal{V})$
- $K = G_u = \{g \in G : gu = u\}$
- $\mathscr{H} = \{B \in \mathrm{End}(\mathbb{C}^{P(\mathcal{V})}) : gB = Bg$ for $\forall g \in K\}$

- Dunkl (1978) decomposed $\mathbb{C}^{P(\mathcal{V})}$ into irreducible $K$-modules, and computed all the spherical functions, i.e., the structure of $\mathscr{H}$ is (essentially) known.

# Grassmann graphs

- $\mathscr{H} = \{B \in \mathrm{End}(\mathbb{C}^{P(\mathcal{V})}) : gB = Bg \text{ for } \forall g \in K\}$ $\longleftarrow$ known
- $K \curvearrowright X = \begin{bmatrix} \mathcal{V} \\ D \end{bmatrix}$
- $\mathscr{H}_X = \{B \in \mathrm{End}(\mathbb{C}^{X}) : gB = Bg \text{ for } \forall g \in K\}$ $\longleftarrow$ known
- $Y_i = \{z \in X : \dim(u \cap z) = n - i\} \ (0 \leqslant i \leqslant n)$
- $K \cdot Y_i = Y_i \implies E_i^* \in \mathscr{H}_X$
- $\boldsymbol{T} \subset \mathscr{H}_X$

### Theorem

- *Every irreducible $\mathscr{H}_X$-module is a thin irreducible $\boldsymbol{T}$-module.*

# Semilattice-type DRGs

done

- $\Gamma$ : a Johnson, Hamming, Grassmann, bilinear forms, or a dual polar graph

(very) hard

- $(\mathscr{P}, \preccurlyeq)$ : the associated semilattice
- $u \in \mathscr{P}$ : rank $n$
- $Y = \{z \in X : u \preccurlyeq z\}$ : a descendent with $w = D - n$, $w^* = n$

**Theorem (BGKM, 2003; T., 2006)**

- *Every descendent of* $\Gamma$ *with* $w^* = n$ *is isomorphic (under* $\mathrm{Aut}\,\Gamma$*) to* $Y$ *above.*

# The bipartite $Q$-polynomial DRGs

- Suppose $\Gamma$ is bipartite.

### Theorem (Caughman, 1999)

- *The structure of* $\boldsymbol{T}(x)$ *depends only on the parameters of* $\Gamma$.

- The dual polar graphs $[D_D(q)]$ and the Hemmeter graphs $\mathrm{Hem}_D(q)$ have the same parameters.

- $Y$ : an edge of $\Gamma$; a descendent with $w = 1$, $w^* = D - 1$

a Delsarte clique

### Problem

- Study $\boldsymbol{T}(Y)$.