A semidefinite programming approach to a cross-intersection problem with measures

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Systems of Lines

Basic problem

- Ω : a finite set
- G : a (simple) graph with $V(G) = \Omega$
- $\alpha(G)$: the independence number of G

$$:= \max \left\{ |U| : U \subset \Omega : \text{independent (i.e., no edge inside)} \right\}$$

Problem

- Find a good upper bound on $\alpha(G)$.

An SDP relaxation

- $\mathbb{R}^{\Omega \times \Omega} = \{$real matrices indexed by $\Omega\}$
- $\mathbb{R}^{\Omega} = \{$real column vectors indexed by $\Omega\}$
- $S\mathbb{R}^{\Omega \times \Omega} = \{$symmetric matrices in $\mathbb{R}^{\Omega \times \Omega}\}$

- $X \succcurlyeq 0 \overset{\text{def}}{\iff} X$: positive semidefinite
- $Y \bullet Z := \operatorname{trace}(Y^{\mathsf{T}} Z)$

- $I \in \mathbb{R}^{\Omega \times \Omega}$: the identity matrix
- $J \in \mathbb{R}^{\Omega \times \Omega}$: the all ones matrix
- $A \in \mathbb{R}^{\Omega \times \Omega}$: the adjacency matrix of G:

$$A_{x,y} = \begin{cases} 1 & \text{if } x \sim y \\ 0 & \text{otherwise} \end{cases}$$

An SDP relaxation

- $U \subset \Omega$: independent
- $\boldsymbol{\varphi} \in \mathbb{R}^{\Omega}$: the characteristic vector of U:

$$\varphi_x = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$

- $X := \frac{1}{|U|} \boldsymbol{\varphi}\boldsymbol{\varphi}^{\mathsf{T}} \in S\mathbb{R}^{\Omega \times \Omega}$
  - $X \succcurlyeq 0$, $\quad X \geqslant 0$ (non-negative)
  - $I \bullet X = \frac{1}{|U|} \boldsymbol{\varphi}^{\mathsf{T}} \boldsymbol{\varphi} = 1$, $\quad A \bullet X = \frac{1}{|U|} \boldsymbol{\varphi}^{\mathsf{T}} A \boldsymbol{\varphi} = 0$
  - $J \bullet X = \frac{1}{|U|} \boldsymbol{\varphi}^{\mathsf{T}} J \boldsymbol{\varphi} = |U|$

An SDP relaxation

- Consider the following SDP problem in primal standard form:

$$\text{(P):}\quad \vartheta' = \max_X \; J \bullet X, \quad X \in S\mathbb{R}^{\Omega\times\Omega},$$
$$I \bullet X = 1,\; A \bullet X = 0,$$
$$X \succcurlyeq 0,\; X \geqslant 0.$$

- Then $|U| \leqslant \vartheta'$.

Remark

- $\vartheta' =$ the strengthening of Lovász's ϑ-function bound due to Schrijver (1979)

- A feasible solution to the dual problem provides an upper bound on $|U|$:

$$\text{(D)}: \quad \vartheta' = \min_{\alpha, \gamma, S, Z} \alpha, \qquad \alpha, \gamma \in \mathbb{R},\ S, Z \in S\mathbb{R}^{\Omega \times \Omega}, \\ \alpha I - J = S + Z + \gamma A, \\ S \succcurlyeq 0,\ Z \geqslant 0.$$

Proof (of weak duality).

$$\begin{aligned} \alpha - J \bullet X &= \alpha I \bullet X - J \bullet X \\ &= S \bullet X + Z \bullet X + \gamma A \bullet X \\ &= S \bullet X + Z \bullet X \\ &\geqslant 0 \end{aligned}$$

□

Delsarte's LP bound (1972, 1973)

- If $I, J, A \in \exists$ Bose–Mesner algebra, then (P), (D) reduce to LP [Schrijver (1979)] !!

Example (Delsarte (1972))

- Bounds on codes in $\mathbb{F}_q^n \longrightarrow$ Hamming scheme $H(n, q)$

Delsarte's LP bound (1972, 1973)

Example (Erdős–Ko–Rado (1961); Wilson (1984))

- $[n] := \{1, 2, \ldots, n\}$
- some conditions on n, k, t
- $U \subseteq \binom{[n]}{k}$: t-intersecting, i.e., $|x \cap y| \geqslant t \ \ (\forall x, y \in U)$
- Then $|U| \leqslant \binom{n-t}{k-t}$.
- $|U| = \binom{n-t}{k-t} \iff \exists z \in \binom{[n]}{t}$ s.t.

$$U = \left\{ x \in \binom{[n]}{k} : z \subset x \right\}$$

$\longrightarrow$ Johnson scheme $J(n, k)$

My motivation

- Bose–Mesner algebra : commutative [SDP $\longrightarrow$ LP]
- Consider cases where the underlying algebras are non-commutative!!

Example (Schrijver (2005); Gijswijt–Schrijver–T. (2006))

- SDP bounds on codes in $\mathbb{F}_q^n$ based on the Terwilliger algebra of $H(n, q)$ [Key idea: $X = X' + X''$ (matrix-cut)]

Example (Bachoc–Vallentin (2008))

- new proof of $k(4) = 24$ using Schrijver's method (originally due to Musin (2008)).

kissing number in $\mathbb{R}^4$

A two-step generalization of the problem

- Ω_1, Ω_2 : non-empty finite sets
- $\widehat{\Omega} := \Omega_1 \sqcup \Omega_2$

- G : a bipartite graph with bipartition $V(G) = \widehat{\Omega} = \Omega_1 \sqcup \Omega_2$
- $U_1 \subset \Omega_1,\, U_2 \subset \Omega_2$: cross-independent in G
  $\stackrel{\text{def}}{\Longleftrightarrow}$ $U_1 \sqcup U_2$: independent in G

- μ_i : a probability measure on Ω_i $\ (i = 1, 2)$

Problem

- Find a good upper bound on $\mu_1(U_1)\mu_2(U_2)$ for cross-independent $U_1 \subset \Omega_1,\, U_2 \subset \Omega_2$.

A two-step generalization of the problem

Example (Pyber (1986); Matsumoto–Tokushige (1989))

- $[n] := \{1, 2, \ldots, n\}$
- some conditions on n, k, ℓ
- $U_1 \in \binom{[n]}{k}$, $U_2 \in \binom{[n]}{\ell}$: cross-intersecting, i.e., $x \cap y \neq \emptyset$ $(\forall x \in U_1,\ \forall y \in U_2)$
- Then $|U_1||U_2| \leqslant \binom{n-1}{k-1}\binom{n-1}{\ell-1}$.
- $|U_1||U_2| = \binom{n-1}{k-1}\binom{n-1}{\ell-1}$
$\iff \exists r \in [n]$ s.t. $U_1 = \left\{x \in \binom{[n]}{k} : r \in x\right\}$, $U_2 = \left\{y \in \binom{[n]}{\ell} : r \in y\right\}$

Remark

- $\exists$SDP-based proof: $\lim_{q \to 1}$ [Suda–T. (2014)]
- Here, we consider a coherent algebra with two fibers.

A generalization of Schrijver's ϑ'

- $\mathbb{R}^{\widehat{\Omega}\times\widehat{\Omega}} = \{$real matrices indexed by $\widehat{\Omega}\}$
- $\mathbb{R}^{\widehat{\Omega}} = \{$real column vectors indexed by $\widehat{\Omega}\}$
- $S\mathbb{R}^{\widehat{\Omega}\times\widehat{\Omega}} = \{$symmetric matrices in $\mathbb{R}^{\widehat{\Omega}\times\widehat{\Omega}}\}$

- $\mathbb{R}^{\Omega_i\times\Omega_j} \subset \mathbb{R}^{\widehat{\Omega}\times\widehat{\Omega}}$, $\mathbb{R}^{\Omega_i} \subset \mathbb{R}^{\widehat{\Omega}}$: defined in the same manner

- $\Delta_i \in \mathbb{R}^{\Omega_i\times\Omega_i}$: the diagonal matrix with

$$(\Delta_i)_{x,x} = \mu_i(\{x\})$$

- $J_{i,j} \in \mathbb{R}^{\Omega_i\times\Omega_j}$: the all ones matrix
- $A = \begin{bmatrix} 0 & A_{1,2} \\ A_{2,1} & 0 \end{bmatrix} \in \mathbb{R}^{\Omega\times\Omega}$: the adjacency matrix of G

A generalization of Schrijver's ϑ'

- $U_1 \subset \Omega_1,\ U_2 \subset \Omega_2$: cross-independent
- $\boldsymbol{\varphi}_i \in \mathbb{R}^{\Omega_i}$: the characteristic vector of $U_i \quad (i = 1, 2)$
- $X := \begin{bmatrix} \frac{1}{\sqrt{\mu_1(U_1)}} \boldsymbol{\varphi}_1 \\ \frac{1}{\sqrt{\mu_2(U_2)}} \boldsymbol{\varphi}_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{\mu_1(U_1)}} \boldsymbol{\varphi}_1 \\ \frac{1}{\sqrt{\mu_2(U_2)}} \boldsymbol{\varphi}_2 \end{bmatrix}^{\mathsf{T}} \in S\mathbb{R}^{\widehat{\Omega} \times \widehat{\Omega}}$
    - $X \succcurlyeq 0, \quad X \geqslant 0$
    - $\begin{bmatrix} \Delta_1 & 0 \\ 0 & 0 \end{bmatrix} \bullet X = \begin{bmatrix} 0 & 0 \\ 0 & \Delta_2 \end{bmatrix} \bullet X = 1, \quad A \bullet X = 0$
    - $\begin{bmatrix} 0 & \frac{1}{2}\Delta_1 J_{1,2} \Delta_2 \\ \frac{1}{2}\Delta_2 J_{2,1} \Delta_1 & 0 \end{bmatrix} \bullet X = \sqrt{\mu_1(U_1)\mu_2(U_2)}$

A generalization of Schrijver's ϑ'

- Consider the following SDP problem in primal standard form:

$$\text{(P):}\quad \widehat{\vartheta}' = \max_{X} \begin{bmatrix} 0 & \frac{1}{2}\Delta_1 J_{1,2}\Delta_2 \\ \frac{1}{2}\Delta_2 J_{2,1}\Delta_1 & 0 \end{bmatrix} \bullet X, \quad X \in S\mathbb{R}^{\widehat{\Omega}\times\widehat{\Omega}},$$

$$\begin{bmatrix} \Delta_1 & 0 \\ 0 & 0 \end{bmatrix} \bullet X = \begin{bmatrix} 0 & 0 \\ 0 & \Delta_2 \end{bmatrix} \bullet X = 1, \quad A \bullet X = 0,$$

$$X \succcurlyeq 0, \ X \geqslant 0.$$

- Then $\sqrt{\mu_1(U_1)\mu_2(U_2)} \leqslant \widehat{\vartheta}'$.

- A feasible solution to the dual problem provides an upper bound on $\sqrt{\mu_1(U_1)\mu_2(U_2)}$:

$$\text{(D):}\quad \widehat{\vartheta}' = \min_{\alpha,\beta,\gamma,S,Z} \ \alpha+\beta, \qquad \alpha,\beta,\gamma\in\mathbb{R},\ S,Z\in S\mathbb{R}^{\Omega\times\Omega},$$

$$\begin{bmatrix} \alpha\Delta_1 & -\frac{1}{2}\Delta_1 J_{1,2}\Delta_2 \\ -\frac{1}{2}\Delta_2 J_{2,1}\Delta_1 & \beta\Delta_2 \end{bmatrix} = S+Z+\gamma A,$$

$$S\succcurlyeq 0,\ Z\geqslant 0.$$

Main result; an application of $\widehat{\vartheta}'$

- $2^{[n]}$: the power set of $[n] := \{1, 2, \ldots, n\}$
- Ω_1, Ω_2 : copies of $2^{[n]}$

- $\boldsymbol{p}, \boldsymbol{q} \in (0,1)^n$
- $\mu_1 = \mu_{\boldsymbol{p}} : 2^{\Omega_1} \to [0,1]$: a product measure on Ω_1:

$$\mu_1(U) := \sum_{x \in U} \prod_{r \in x} p_r \prod_{s \in [n] \setminus x} \Big(1 - p_s\Big) \qquad (U \subset \Omega_1)$$

[Note: $\mu_1(\Omega_1) = \big(p_1 + (1 - p_1)\big) \cdots \big(p_n + (1 - p_n)\big) = 1$]

- $\mu_2 = \mu_{\boldsymbol{q}} : 2^{\Omega_2} \to [0,1]$: a product measure on Ω_2

Main result; an application of $\widehat{\vartheta}'$

Theorem (Suda–T.–Tokushige (2015))

- *Suppose*
  - $p_1 = \max\{p_r : r \in [n]\}, \quad q_1 = \max\{q_r : r \in [n]\}$
  - $p_r, q_r \leqslant \frac{1}{2} \ (\forall r \geqslant 2)$.
- $U_1 \subset \Omega_1$, $U_2 \subset \Omega_2$ *: cross-intersecting*
- *Then* $\mu_1(U_1)\mu_2(U_2) \leqslant p_1 q_1$.
- *If* $\mu_1(U_1)\mu_2(U_2) = p_1 q_1$ *then* $\exists r \in w$ *s.t.*

$$U_1 = \big\{x \in \Omega_1 : r \in x\big\}, \quad U_2 = \big\{y \in \Omega_2 : r \in y\big\}$$

unless $p_1 = q_1 = \frac{1}{2}$ *and* $|w| \geqslant 3$, *where*

$$w = w_{\boldsymbol{p},\boldsymbol{q}} := \big\{r \in [n] : (p_r, q_r) = (p_1, q_1)\big\}.$$

Main result; an application of $\widehat{\vartheta}'$

Remark

- The theorem generalizes and strengthens a result of Fishburn–Frankl–Freed–Lagarias–Odlyzko (1986) for intersecting families.
- Partial results were obtained previously:
  - Tokushige (2010) : $\frac{1}{2} > p_1 = \cdots = p_n,\ \frac{1}{2} > q_1 = \cdots = q_n$
  - Borg (2012) : $\frac{1}{2} \geqslant p_1 \geqslant \cdots \geqslant p_n,\ \frac{1}{2} \geqslant q_1 \geqslant \cdots \geqslant q_n$ (not precise)

How the proof proceeds [see arXiv:1504.00135 for the details]

- In fact, the SDP method works only when $p_1, q_1 \leqslant \frac{1}{2}$, and invokes an idea of Friedgut (2008):
  - Find a "nice" feasible solution when $n = 1$;
  - Construct feasible solutions for general n by taking "tensor products".
- When $p_1 > \frac{1}{2}$ or $q_1 > \frac{1}{2}$, the proof is reduced to the above case by considering

$$\tilde{\boldsymbol{p}} := \Big(\max_{r \geqslant 2} p_r,\ p_2, \ldots, p_n\Big), \quad \tilde{\boldsymbol{q}} := \Big(\max_{r \geqslant 2} q_r,\ q_2, \ldots, q_n\Big),$$

which is an idea of Fishburn et al. (1986).