Current progress in the Delsarte theory

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The Delsarte Theory (1973)

- Philippe Delsarte,
  An algebraic approach to the association schemes of coding theory,
  Philips Res. Rep. Suppl. No. 10 (1973).

Warning

- a very quick survey!!
- not at all comprehensive!!

Story in a nutshell

- LP $\longrightarrow$ SDP
- adjacency algebra $\longrightarrow$ Terwilliger algebra
- single code/design $\longrightarrow$ two (or more) codes/designs

Notation

- $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$: a symmetric association scheme
- $A_0, \ldots, A_d$: the adjacency matrices
- $\mathfrak{A} = \langle A_0, \ldots, A_d \rangle$: the adjacency algebra (over $\mathbb{C}$)
- $E_0, \ldots, E_d$: the primitive idempotents of $\mathfrak{A}$
- P, Q : the first and second eigenmatrices, i.e.,

$$A_i = \sum_{j=0}^{d} P_{j,i} \, E_j, \quad E_i = \frac{1}{|X|} \sum_{j=0}^{d} Q_{j,i} \, A_j$$

- k_i : the valency (or degree) of (X, R_i) $(0 \leqslant i \leqslant d)$
- $m_i = \operatorname{rank} E_i = \operatorname{trace} E_i$ $(0 \leqslant i \leqslant d)$

$\mathscr{D}$-codes

- $\mathscr{D} \subset \{1, 2, \ldots, d\}$
- $C \subset X$: a $\mathscr{D}$-code

  $\overset{\text{def}}{\Longleftrightarrow}$ C : an independent set of the graph $\left(X, \bigcup_{i \in \mathscr{D}} R_i\right)$

  $\Longleftrightarrow$ $(C \times C) \cap R_i = \emptyset \quad (i \in \mathscr{D})$

Example

- $\mathfrak{X} = H(d, q)$: the Hamming scheme
- $X = \mathbb{F}_q^d$
- C : e-error correcting $\Longleftrightarrow$ C : a $\{1, \ldots, 2e\}$-code

$\mathscr{D}$-codes

- $\chi = \chi_C \in \mathbb{R}^X$: the (column) characteristic vector of C, i.e.,

$$\chi_x = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases} \qquad (x \in X)$$

- $\chi^{\mathsf{T}} A_i \chi = \left|(C \times C) \cap R_i\right|$

- C : a $\mathscr{D}$-code $\iff \chi^{\mathsf{T}} A_i \chi = 0 \ \ (i \in \mathscr{D})$

$\mathscr{D}$-designs

- $\mathscr{D} \subset \{1, 2, \ldots, d\}$
- $C \subset X$: a $\mathscr{D}$-design $\overset{\text{def}}{\iff} \chi^{\mathsf{T}} E_i \chi = 0 \;\; (i \in \mathscr{D})$

Theorem (Delsarte, 1973)

- $\mathfrak{X} = J(v, d)$: *the Johnson scheme*
- C : *a* t-(v, d, λ) *design (for some* λ*)* $\iff$ C : *a* $\{1, \ldots, t\}$*-design*

Theorem (Delsarte, 1973)

- $\mathfrak{X} = H(d, q)$: *the Hamming scheme*
- C : *an orthogonal array of strength* t $\iff$ C : *a* $\{1, \ldots, t\}$*-design*

The LP bound for $\mathscr{D}$-codes

- $C \subset X$: a $\mathscr{D}$-code
- $e_i := \dfrac{\chi^{\mathsf{T}} A_i \chi}{|C|} \quad (0 \leqslant i \leqslant d)$
- $\boldsymbol{e} = (e_0, \ldots, e_d)$: the inner distribution of C

- $e_i = 0 \quad (i \in \mathscr{D})$
- $e_0 = 1$

$\because\ A_0 = I$

The LP bound for $\mathscr{D}$-codes

- View the e_i as real variables !!

Theorem (Delsarte, 1973)

- *Consider the following LP problem:*

$$\texttt{maximize} \quad \vartheta' = \sum_{i=0}^{d} e_i \quad \leftarrow |C|$$

$\texttt{subject to}$
- $e_0 = 1$
- $e_i = 0 \ \texttt{for} \ i \in \mathscr{D}$
- $e_i \geqslant 0 \ \texttt{for} \ i \in \{1, 2, \dots, d\} \backslash \mathscr{D}$
- $(\boldsymbol{e}Q)_i \geqslant 0 \ \texttt{for} \ i \in \{1, 2, \dots, d\}$ $\quad \leftarrow \chi^{\mathsf{T}} E_i \chi \geqslant 0$

- *If C is a $\mathscr{D}$-code, then $|C| \leqslant \vartheta'$.*

Codes in P-polynomial schemes

- $\mathfrak{X}$: P-polynomial
  $\stackrel{\text{def}}{\Longleftrightarrow}$ $\exists v_0(\xi), \ldots, v_d(\xi) \in \mathbb{R}[\xi], \ \exists \theta_0, \ldots, \theta_d \in \mathbb{R}$ s.t.
  - $\deg v_i(\xi) = i \quad (0 \leqslant i \leqslant d)$
  - $P_{j,i} = v_i(\theta_j) \quad (0 \leqslant i, j \leqslant d)$

- Suppose $\mathfrak{X}$ is P-polynomial.
- $\delta := \min\{i \neq 0 : \chi^{\mathsf{T}} A_i \chi \neq 0\}$: the minimum distance of C
- $s^* := \left|\{i \neq 0 : \chi^{\mathsf{T}} E_i \chi \neq 0\}\right|$: the dual degree of C

Theorem (Delsarte, 1973)

- $\delta \leqslant 2s^* + 1$
- *If $\delta \geqslant 2s^* - 1$, then C is completely regular.*

Codes in P-polynomial schemes

- Recall $\delta \leqslant 2s^* + 1$.
- $|C| \leqslant \dfrac{|X|}{k_0 + \cdots + k_{\lfloor \frac{\delta-1}{2} \rfloor}}$: the sphere-packing bound

  ← the volume of a ball of radius $\lfloor \frac{\delta-1}{2} \rfloor$
- C : perfect $\overset{\text{def}}{\Longleftrightarrow}$ "$=$" holds above
- $L_j(\xi) := v_0(\xi) + \cdots + v_j(\xi)$: the Lloyd polynomial of degree j

Theorem (Delsarte, 1973)

- $\delta = 2s^* + 1 \iff C$ *: perfect* $\implies L_{s^*}(\xi)$ *has* s^* *simple roots in* $\{\theta_0, \ldots, \theta_d\}$.

Designs in Q-polynomial schemes

- $\mathfrak{X}$: Q-polynomial
$\overset{\text{def}}{\Longleftrightarrow}$ $\exists v_0^*(\xi), \ldots, v_d^*(\xi) \in \mathbb{R}[\xi]$, $\exists \theta_0^*, \ldots, \theta_d^* \in \mathbb{R}$ s.t.
  - $\deg v_i^*(\xi) = i \quad (0 \leqslant i \leqslant d)$
  - $Q_{j,i} = v_i^*(\theta_j^*) \quad (0 \leqslant i, j \leqslant d)$

- Suppose $\mathfrak{X}$ is Q-polynomial.
- $\delta^* := \min\{i \neq 0 : \chi^{\mathsf{T}} E_i \chi \neq 0\}$: the dual distance of C
- $s := \left|\{i \neq 0 : \chi^{\mathsf{T}} A_i \chi \neq 0\}\right|$: the degree of C

Theorem (Delsarte, 1973)

- $\delta^* \leqslant 2s + 1$
- *If $\delta^* \geqslant 2s - 1$, then C induces a Q-polynomial scheme.*

Designs in Q-polynomial schemes

- Recall $\delta^* \leqslant 2s+1$.
- $|C| \geqslant m_0 + \cdots + m_{\lfloor \frac{\delta^*-1}{2} \rfloor}$: the Fisher type inequality
- C : tight $\overset{\text{def}}{\Longleftrightarrow}$ "$=$" holds above
- $W_j(\xi) := v_0^*(\xi) + \cdots + v_j^*(\xi)$: the Wilson polynomial of degree j

Theorem (Delsarte, 1973)

- $\delta^* = 2s+1 \iff C$ *: tight* $\implies W_s(\xi)$ *has* s *simple roots in* $\{\theta_0^*, \ldots, \theta_d^*\}$.

Width and dual width (2003)

- Suppose $\mathfrak{X}$ is P-polynomial.

Theorem (Delsarte, 1973)

- $\delta \leqslant 2s^* + 1$
- *If $\delta \geqslant 2s^* - 1$, then C is completely regular.*

- $w := \max\{i : \chi^{\mathsf{T}} A_i \chi \neq 0\}$: the width of C

Theorem (Brouwer–Godsil–Koolen–Martin, 2003)

- $w \geqslant d - s^*$
- *If $w = d - s^*$, then C is completely regular.*

$\# = s$

$$\boldsymbol{e} = (1, 0, \ldots, 0, *, 0, \ldots, 0, *, \ldots, *, 0, \ldots, 0)$$

δ w

Width and dual width (2003)

- Suppose $\mathfrak{X}$ is Q-polynomial.

Theorem (Delsarte, 1973)

- $\delta^* \leqslant 2s + 1$
- *If $\delta^* \geqslant 2s - 1$, then C induces a Q-polynomial scheme.*

- $w^* := \max\{i : \chi^{\mathsf{T}} E_i \chi \neq 0\}$: the dual width of C

Theorem (Brouwer–Godsil–Koolen–Martin, 2003)

- $w^* \geqslant d - s$
- *If $w^* = d - s$, then C induces a Q-polynomial scheme.*

$\# = s^*$

$$\boldsymbol{e}Q = (|C|, 0, \ldots, 0, *, 0, \ldots, 0, *, \ldots, *, 0, \ldots, 0)$$

δ^* w^*

Width and dual width (2003)

- Suppose $\mathfrak{X}$ is both P-polynomial & Q-polynomial.
- $w \geqslant d - s^*, \ w^* \geqslant s^* \implies w + w^* \geqslant d$
- $w^* \geqslant d - s, \ w \geqslant s \implies w + w^* \geqslant d$

Corollary (Brouwer–Godsil–Koolen–Martin, 2003)

- $w + w^* \geqslant d$
- *If $w + w^* = d$, then C is completely regular and induces a Q-polynomial scheme.*

- C : a descendent $\stackrel{\text{def}}{\iff} w + w^* = d$

The Erdős–Ko–Rado Theorem (1961)

Theorem (Erdős–Ko–Rado, 1961; Wilson, 1984)

- $v, d, t \in \mathbb{N}$ *s.t.* $v > (t+1)(d-t+1)$
- $[v] := \{1, 2, \ldots, v\}$
- $C \subset \binom{[v]}{d}$: *t-intersecting*, *i.e.*, $|x \cap y| \geqslant t \quad (x, y \in C)$
- *Then* $|C| \leqslant \binom{v-t}{d-t}$. $\longleftarrow$ *true when* $v \geqslant (t+1)(d-t+1)$
- "$=$" $\iff \exists z \in \binom{[v]}{t}$ s.t. $C = \left\{ x \in \binom{[v]}{d} : z \subset x \right\} \cong J(v-t, d-t)$

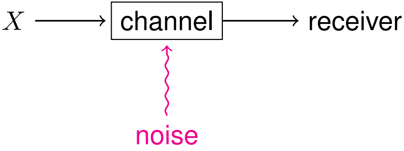
"the" descendents in $J(v, d)$

Remark

- Delsarte's LP + BGKM led to the EKR theorems for:
  - Grassmann schemes (T., 2006);
  - bilinear forms schemes (T., 2006);
  - twisted Grassmann schemes (T., 2012).

Shannon's zero error capacity (1956)

- X : a finite set $=$ a set of letters



- Define the graph $G = (X, R)$ as follows:

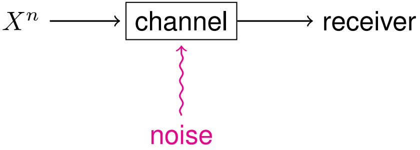
$$x \sim y \overset{\text{def}}{\iff} x, y \text{ can be “confused”} \quad (x, y \in X)$$

- Find the independence number $\alpha(G)$ of G !!

a good upper bound on

Shannon's zero error capacity (1956)

- Send words of length n:



- $\boldsymbol{x} = (x_1, \ldots, x_n), \boldsymbol{y} = (y_1, \ldots, y_n)$ can be sent without confusion
$\iff$ $x_i \not\sim y_i$ for some i

non-adjacency in the strong product $G \boxtimes \cdots \boxtimes G = G^{\boxtimes n}$

- $c(G) := \sup_n \sqrt[n]{\alpha(G^{\boxtimes n})}$: the Shannon capacity of G

Lovász's ϑ-function bound for $c(G)$ (1979)

- Recall $c(G) = \sup_n \sqrt[n]{\alpha(G^{\boxtimes n})}$.

Theorem (Lovász, 1979)

- *Consider the following SDP problem:*

  maximize $\vartheta = \operatorname{trace} MJ$ ← *all-ones matrix*

  subject to
  - $\operatorname{trace} M = 1$
  - $M_{x,y} = 0$ if $x \sim y$
  - M : positive semidefinite

- *Then* $c(G) \leqslant \vartheta$.

Remark

- $\vartheta = \vartheta(G)$ satisfies $\alpha(G) \leqslant \vartheta(G)$ and $\vartheta(G^{\boxtimes n}) = \vartheta(G)^n$.

Lovász's ϑ-function bound for $c(G)$ (1979)

Lemma (Schrijver, 1979)

- *Consider the following SDP problem:*

  maximize $\vartheta' = \operatorname{trace} MJ$

  subject to
  - $\operatorname{trace} M = 1$
  - $M_{x,y} = 0$ if $x \sim y$
  - M : positive semidefinite & non-negative

- *Then* $\alpha(G) \leqslant \vartheta' (\leqslant \vartheta)$.

Theorem (Schrijver, 1979)

- *If* $G = \left(X, \bigcup_{i \in \mathscr{D}} R_i\right)$, *then* ϑ' *coincides with Delsarte's LP bound for* $\mathscr{D}$*-codes.*

idea: project M to $\mathfrak{A}$!!

The Terwilliger algebra (1992)

- We will consider the hypercube $H(d, 2)$ as an illustrative example!!

- $X = \mathbb{F}_2^d$ $=$ the power set of $[d] := \{1, 2, \ldots, d\}$
- $E_0^*, \ldots, E_d^*$: the dual idempotents of $H(d, 2)$, i.e.,

$$(E_i^*)_{x,y} = \begin{cases} 1 & \text{if } x = y \in \binom{[d]}{i} \\ 0 & \text{otherwise} \end{cases} \qquad (x, y \in X)$$

- $\mathfrak{T} = \langle E_i^* A_j E_h^* : 0 \leqslant i, j, h \leqslant d \rangle$: the Terwilliger algebra

Remark

- $E_i^* \mathfrak{T} E_i^* =$ the adjacency algebra of $J(d, i)$ $(0 \leqslant i \leqslant d)$

Schrijver's SDP bound for $\mathscr{D}$-codes (2005)

- Recall Schrijver's problem for $G = \left(X, \bigcup_{i\in\mathscr{D}} R_i\right)$:

maximize $\vartheta' = \operatorname{trace} MJ$

subject to
- $M \in \mathfrak{A}$
- $\operatorname{trace} M = 1$
- $M_{x,y} = 0$ if $x \sim y$
- M : positive semidefinite & non-negative

Schrijver's SDP bound for $\mathscr{D}$-codes (2005)

- Schrijver (2005) strengthened the problem as follows:

$$\texttt{maximize}\quad \vartheta'_+ = \operatorname{trace} MJ$$

$\texttt{subject to}$
- $M = M_1 + M_2,\ \ M_1, M_2 \in \mathfrak{T}$
- $\operatorname{trace} M = 1$
- $M_{x,y} = 0$ if $x \sim y$
- M_1, M_2 : positive semidefinite
  & non-negative
  & extra conditions

Theorem (Schrijver, 2005)

- *If C is a $\mathscr{D}$-code, then $|C| \leqslant \vartheta'_+ (\leqslant \vartheta')$.*

Schrijver's SDP bound for $\mathscr{D}$-codes (2005)

Remark

- The bound ϑ'_+ was formulated for some other cases:
  - $J(v, d)$ (Schrijver, 2005);
  - $H(d, q)$ (Gijswijt–Schrijver–T., 2006);
  - $\overline{H(d, 2)}$ (Hou–Hou–Gao–Yu, 2018+).

The primary $\mathfrak{T}$-module

- $\chi_i = \chi_{\binom{[d]}{i}}$: the characteristic vector of $\binom{[d]}{i}$ $(0 \leqslant i \leqslant d)$
- $\langle \chi_0, \ldots, \chi_d \rangle$: the primary $\mathfrak{T}$-module

Relative t-designs (1977)

- C : a t-design $\overset{\text{def}}{\iff}$ C : a $\{1, \ldots, t\}$-design
  $\iff \chi^{\mathsf{T}} E_i \chi = 0 \ \ (1 \leqslant i \leqslant t)$
  $\iff E_i \chi = 0 \ \ (1 \leqslant i \leqslant t)$

- C : a relative t-design $\overset{\text{def}}{\iff}$ $E_i \chi \in \langle E_i \chi_0 \rangle \ \ (1 \leqslant i \leqslant t)$ ($\subset \langle \chi_0, \ldots, \chi_d \rangle$)

Theorem (Delsarte, 1977)

- *C : a relative t-design $\iff$ C : a regular t-wise balanced design*

Remark

- Relative t-designs have been actively studied:

  B.–B., 2012; B.–B.–B., 2014; Li–B.–B., 2014; B.–B.–Suda–T., 2015;
  B.–B.–Zhu, 2015; Yue–Hou–Gao, 2015; Zhu–B.–B., 2016; B.–B.–Zhu, 2017;

Relative t-designs (1977)

- Suppose C is a relative 2ε-design.
- $S := \left\{ i : C \cap \binom{[d]}{i} \neq \emptyset \right\}$

heavily uses $\mathfrak{T}$

Theorem (Xiang, 2012; Bannai–Bannai–Suda–T., 2015)

- *If* $S \subset \{\varepsilon, \ldots, d-\varepsilon\}$, *then* $|C| \geqslant m_\varepsilon + m_{\varepsilon-1} + \cdots + m_{\varepsilon-|S|+1}$.

- C : tight $\overset{\text{def}}{\iff}$ "$=$" holds above

Relative t-designs (1977)

- Recall $S = \left\{ i : C \cap \binom{[d]}{i} \neq \emptyset \right\}$.

Theorem (Bannai–Bannai–T.–Zhu, 2018+)

- *Suppose* $S = \{i, j\}$, *where* $\varepsilon \leqslant i < j \leqslant d - \varepsilon$.
- *If* C *is tight, i.e.,* $|C| = m_\varepsilon + m_{\varepsilon-1}$, *then* C *induces a* *coherent configuration* *with two fibers, and the* *Hahn polynomial*

$$Q_\varepsilon(\xi) = {}_3F_2\left(\begin{matrix} -\xi, -\varepsilon, \varepsilon - d - 1 \\ j - d, -i \end{matrix} \middle| 1 \right) \in \mathbb{R}[\xi]$$

has ε *simple roots in* $\{0, 1, \ldots, \min\{i, d - j\}\}$.

The cross-intersecting EKR theorem (1986)

Theorem (Pyber, 1986; Matsumoto–Tokushige, 1989)

- $d > 2i, 2j$
- $C \subset \binom{[d]}{i}, D \subset \binom{[d]}{j}$: cross-intersecting, i.e., $x \cap y \neq \emptyset$ $(x \in C, y \in D)$
- *Then* $|C||D| \leqslant \binom{d-1}{i-1}\binom{d-1}{j-1}$. $\longleftarrow$ *true when* $d \geqslant 2i, 2j$
- "=" $\iff$ $\exists \ell \in [n]$ s.t. $C = \left\{x \in \binom{[d]}{i} : \ell \in x\right\}$, $D = \left\{y \in \binom{[d]}{j} : \ell \in y\right\}$

The cross-intersecting EKR theorem (1986)

Remark

- $\exists$SDP-based proof $\lim_{q \to 1}$ [Suda–T., 2014]:

$$\texttt{maximize} \quad \ddot{\vartheta}' = \operatorname{trace} M_{i,j} J$$

$\binom{d}{j} \times \binom{d}{i}$ all-ones matrix

$\texttt{subject to}$

- $M = \left[\begin{array}{c|c} M_{i,i} & M_{i,j} \\ \hline M_{j,i} & M_{j,j} \end{array}\right] \in (E_i^* + E_j^*)\,\mathfrak{T}\,(E_i^* + E_j^*)$
- $\operatorname{trace} M_{i,i} = \operatorname{trace} M_{j,j} = 1$
- $(M_{i,j})_{x,y} = 0$ if $x \cap y = \emptyset$
- M : positive semidefinite & non-negative

- This method was explored further (Suda–T.–Tokushige, 2017).

Shannon capacity for two channels

- $G = (X, R)$: a finite simple graph
- $X = X_+ \sqcup X_-$
- $\ddot{\alpha}(G) := \max\left\{ \sqrt{|C_+||C_-|} : C_\pm \subset X_\pm,\ C_+ \cup C_- \text{ is independent} \right\}$

Lemma (Iwabuchi–T., 2018+)

- *Consider the following SDP problem:*

$$\texttt{maximize}\quad \ddot{\vartheta} = \operatorname{trace} M_{+-} J$$

$|X_-| \times |X_+|$ *all-ones matrix*

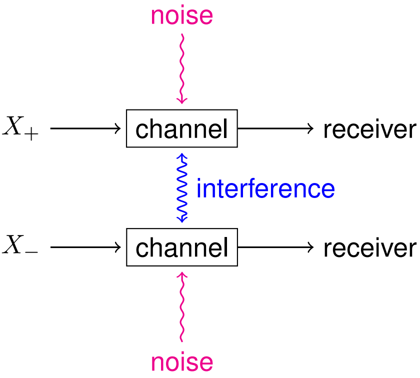
$$\texttt{subject to}\quad \bullet\ M = \left[\begin{array}{c|c} M_{+-} & M_{+-} \\ \hline M_{-+} & M_{--} \end{array}\right]$$

- $\operatorname{trace} M_{++} = \operatorname{trace} M_{--} = 1$
- $M_{x,y} = 0 \texttt{ if } x \sim y$
- M : $\texttt{positive semidefinite}$

- *Then* $\ddot{\alpha}(G) \leqslant \ddot{\vartheta}$.

Shannon capacity for two channels

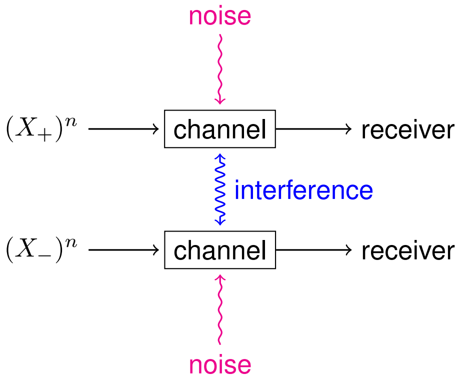
- X_+, X_- : two sets of letters



- $\ddot{\alpha}(G) \longleftrightarrow$ the average rate of the two channels

Shannon capacity for two channels

- Send two sets of words of length n:



Shannon capacity for two channels

- Define the graph $G^{\ddot{\boxtimes} n}$ with vertex set $(X_+)^n \sqcup (X_-)^n$ as follows:

$$\boldsymbol{x} = (x_1, \ldots, x_n) \not\sim \boldsymbol{y} = (y_1, \ldots, y_n)$$

$$\overset{\text{def}}{\iff} \begin{cases} x_i \not\sim y_i \text{ for some } i & \text{if } \operatorname{sgn} \boldsymbol{x} = \operatorname{sgn} \boldsymbol{y} \\ x_i \not\sim y_i \text{ for all } i & \text{if } \operatorname{sgn} \boldsymbol{x} \neq \operatorname{sgn} \boldsymbol{y} \end{cases}$$

(X_+: $\mathrm{sgn} = +$; X_-: $\mathrm{sgn} = -$)

Lemma (Iwabuchi–T., 2018+)

- $\ddot{\vartheta} = \ddot{\vartheta}(G)$ *satisfies* $\ddot{\vartheta}\big(G^{\ddot{\boxtimes} n}\big) = \ddot{\vartheta}(G)^n$.

- $\ddot{c}(G) := \sup_n \sqrt[n]{\ddot{\alpha}\big(G^{\ddot{\boxtimes} n}\big)}$

Theorem (Iwabuchi–T., 2018+)

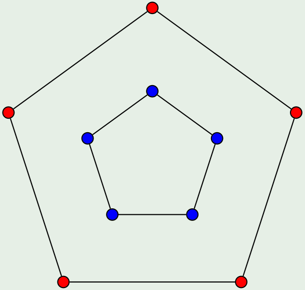
- $\ddot{c}(G) \leqslant \ddot{\vartheta}(G)$

Shannon capacity for two channels

Theorem (Lovász, 1979)

- $c(C_5) = \vartheta(C_5) = \sqrt{5}$

Example



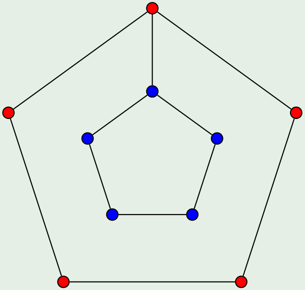
$\ddot{c}(G) = \ddot{\vartheta}(G) = \sqrt{5}$

Shannon capacity for two channels

Theorem (Lovász, 1979)

- $c(C_5) = \vartheta(C_5) = \sqrt{5}$

Example



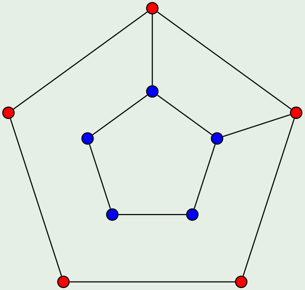
$$\sqrt{2\sqrt{5}} \leqslant \ddot{c}(G) \leqslant \ddot{\vartheta}(G) = \sqrt{5}$$

Shannon capacity for two channels

Theorem (Lovász, 1979)

- $c(C_5) = \vartheta(C_5) = \sqrt{5}$

Example



$$\sqrt{2\sqrt{5}} \leqslant \ddot{c}(G) \leqslant \ddot{\vartheta}(G) = \sqrt{5}$$

Shannon capacity for two channels

Theorem (Lovász, 1979)

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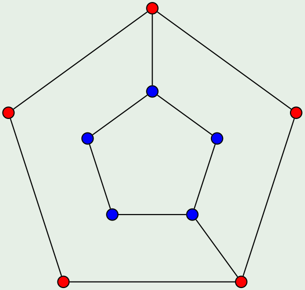
Example



$$\sqrt{2\sqrt{5}} \leqslant \ddot{c}(G) \leqslant \ddot{\vartheta}(G) = \sqrt{5}$$

Shannon capacity for two channels

Theorem (Lovász, 1979)

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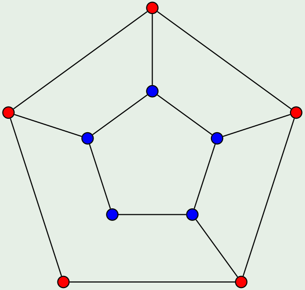
Example



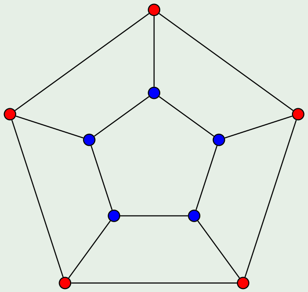
$$2 \leqslant \ddot{c}(G) \leqslant \ddot{\vartheta}(G) = \sqrt{5}$$

Shannon capacity for two channels

Theorem (Lovász, 1979)

- $c(C_5) = \vartheta(C_5) = \sqrt{5}$

Example



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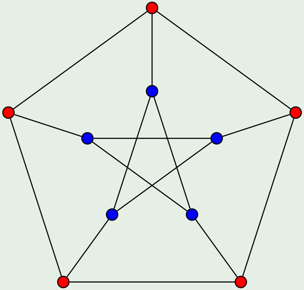
- $c(C_5) = \vartheta(C_5) = \sqrt{5}$

Example



$\ddot{c}(G) = \ddot{\vartheta}(G) = 2$