Bounding the burning number of some distance-regular graphs

Hajime Tanaka
(joint work with Norihide Tokushige)
Research Center for Pure and Applied Mathematics
Graduate School of Information Sciences
Tohoku University

October 16, 2025
Algebraic Graph Theory and Applications
BIRS 25w5432, Banff, Canada

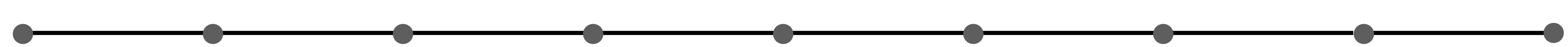
Norihide Tokushige (U. of the Ryukyus)

- He is an expert on Erdős-Ko-Rado-type theorems.
- I have written four papers with him:

1. A semidefinite programming approach to a cross-intersection problem with measures, 2017 (also with Sho Suda).
2. Extremal problems for intersecting families of subspaces with a measure, 2025.
3. A semidefinite programming approach to cross 2-intersecting families, arXiv:2503.14844.
4. Burning numbers via eigenpolytopes — Hamming graphs, Johnson graphs, and halved cubes, arXiv:2508.17559.

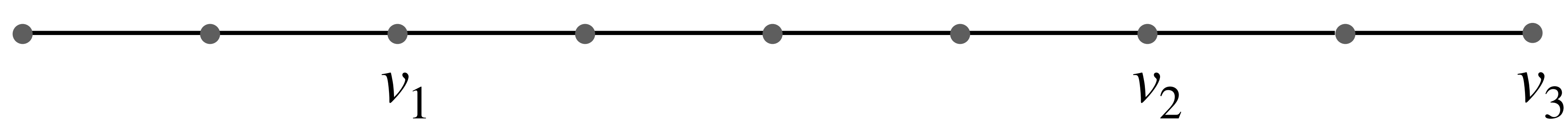
Alon's transmitting problem (1992)

- $G = (V, E)$: a finite connected graph
- A **sender** prepares a sequence of vertices $v_1, v_2, \ldots \in V$, called a **burning sequence**.
- The sender sends a message to v_i at **round** i.
- Every vertex that received the message transmits it to its neighbors at the next round.
- The **burning number** $b(G)$: the minimum number of rounds (over all burning sequences) so that all vertices receive the message.



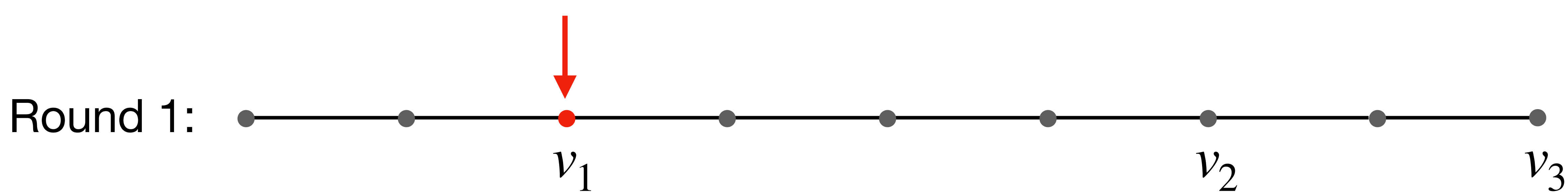
Alon's transmitting problem (1992)

- $G = (V, E)$: a finite connected graph
- A **sender** prepares a sequence of vertices $v_1, v_2, \ldots \in V$, called a **burning sequence**.
- The sender sends a message to v_i at **round** i.
- Every vertex that received the message transmits it to its neighbors at the next round.
- The **burning number** $b(G)$: the minimum number of rounds (over all burning sequences) so that all vertices receive the message.



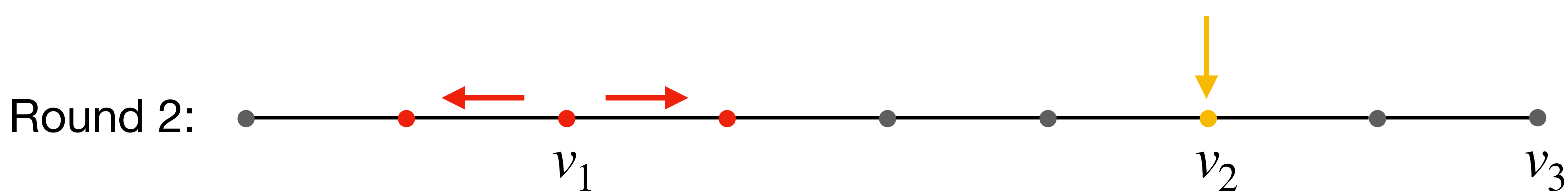
Alon's transmitting problem (1992)

- $G = (V, E)$: a finite connected graph
- A **sender** prepares a sequence of vertices $v_1, v_2, \ldots \in V$, called a **burning sequence**.
- The sender sends a message to v_i at **round** i.
- Every vertex that received the message transmits it to its neighbors at the next round.
- The **burning number** $b(G)$: the minimum number of rounds (over all burning sequences) so that all vertices receive the message.



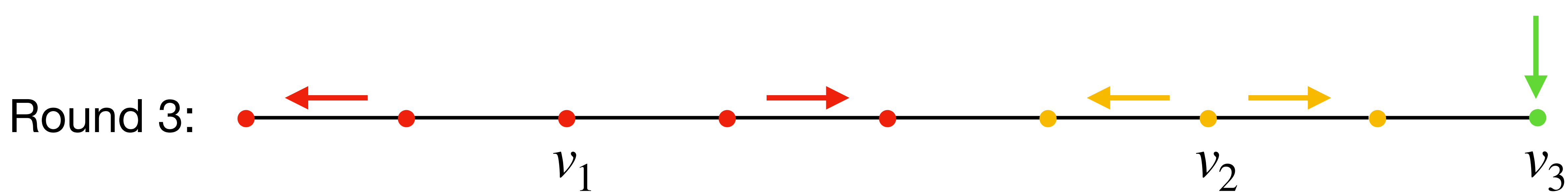
Alon's transmitting problem (1992)

- $G = (V, E)$: a finite connected graph
- A **sender** prepares a sequence of vertices $v_1, v_2, \ldots \in V$, called a **burning sequence**.
- The sender sends a message to v_i at **round** i.
- Every vertex that received the message transmits it to its neighbors at the next round.
- The **burning number** $b(G)$: the minimum number of rounds (over all burning sequences) so that all vertices receive the message.



Alon's transmitting problem (1992)

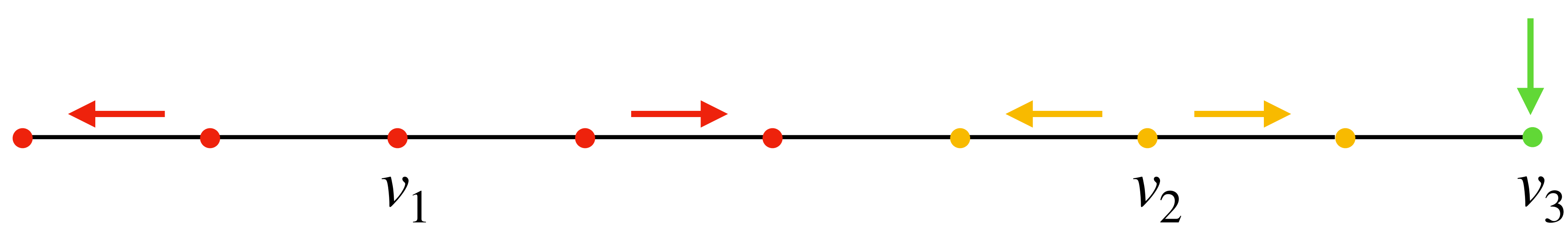
- $G = (V, E)$: a finite connected graph
- A **sender** prepares a sequence of vertices $v_1, v_2, \ldots \in V$, called a **burning sequence**.
- The sender sends a message to v_i at **round** i.
- Every vertex that received the message transmits it to its neighbors at the next round.
- The **burning number** $b(G)$: the minimum number of rounds (over all burning sequences) so that all vertices receive the message.



Alon’s transmitting problem (1992)

- $G = (V, E)$: a finite connected graph
- A **sender** prepares a sequence of vertices $v_1, v_2, \ldots \in V$, called a **burning sequence**.
- The sender sends a message to v_i at **round** i.
- Every vertex that received the message transmits it to its neighbors at the next round.
- The **burning number** $b(G)$: the minimum number of rounds (over all burning sequences) so that all vertices receive the message.

Round 3:

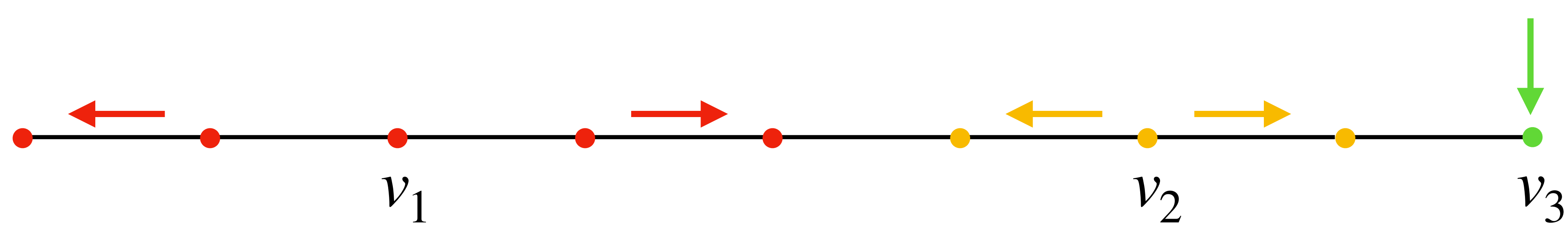


$b(P_9) = 3$

Alon’s transmitting problem (1992)

- $G = (V, E)$: a finite connected graph
- A **sender** prepares a sequence of vertices $v_1, v_2, \ldots \in V$, called a **burning sequence**.
- The sender sends a message to v_i at **round** i.
- Every vertex that received the message transmits it to its neighbors at the next round.
- The **burning number** $b(G)$: the minimum number of rounds (over all burning sequences) so that all vertices receive the message.

Round 3:



$b(P_9) = 3$

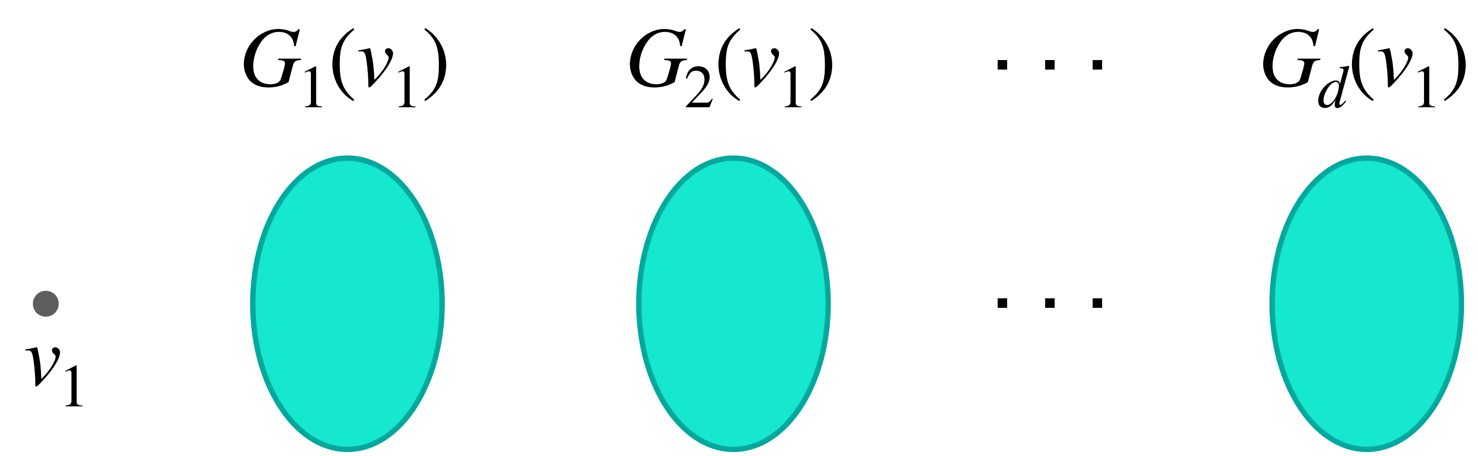
- More generally, $b(P_n) = b(C_n) = \lceil n^{1/2} \rceil$.

path cycle

The burning number $b(G)$

- ∂ : the path-length distance on V
- $G_i(v) = \{u \in V : \partial(u, v) = i\} \quad (v \in V,\ i = 0,1,2,\ldots)$
- $d = \mathrm{diam}(G)$: the diameter of G

Lemma. $b(G) \leqslant d + 1.$

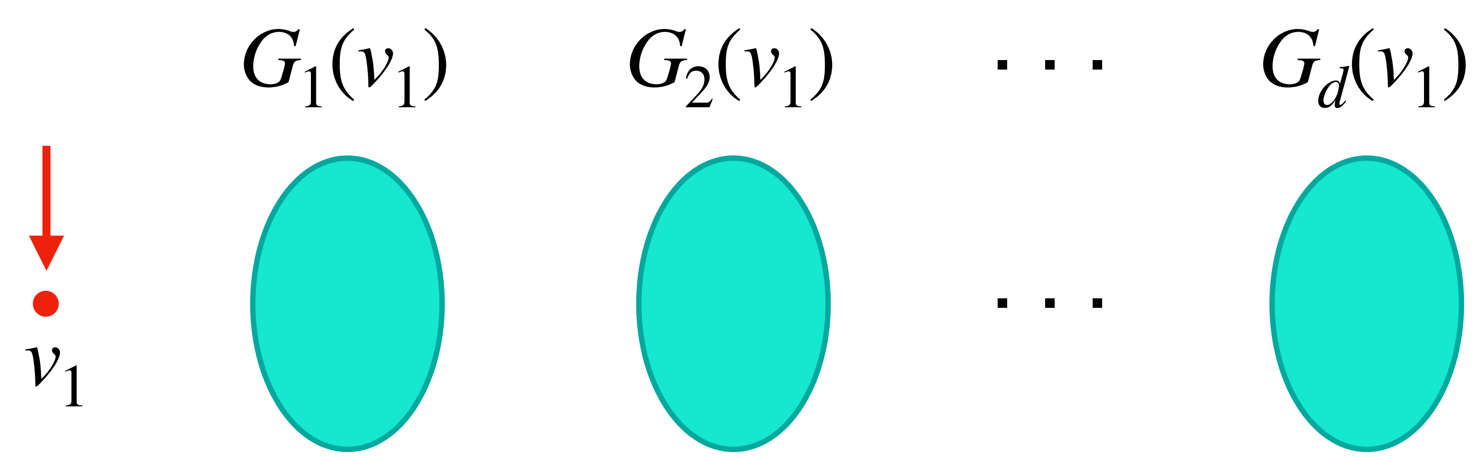


The burning number $b(G)$

- ∂ : the path-length distance on V
- $G_i(v) = \{u \in V : \partial(u, v) = i\} \quad (v \in V,\ i = 0,1,2,\ldots)$
- $d = \operatorname{diam}(G)$: the diameter of G

Lemma. $b(G) \leqslant d + 1.$

Round 1:

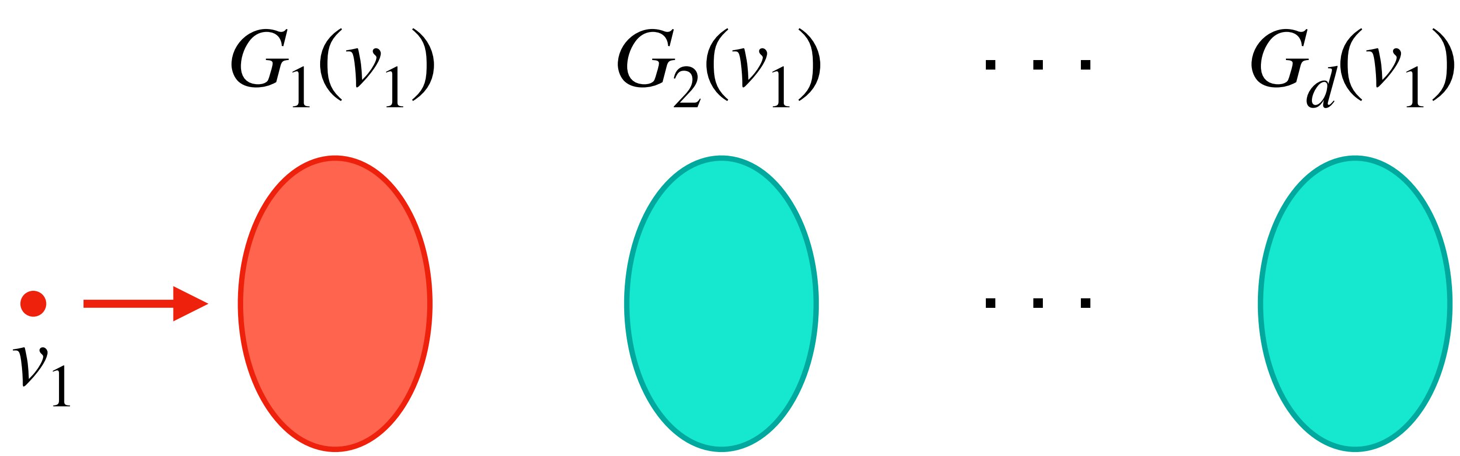


The burning number $b(G)$

- ∂ : the path-length distance on V
- $G_i(v) = \{u \in V : \partial(u, v) = i\} \quad (v \in V,\ i = 0,1,2,\ldots)$
- $d = \mathrm{diam}(G)$: the diameter of G

Lemma. $b(G) \leqslant d + 1.$

Round 2:

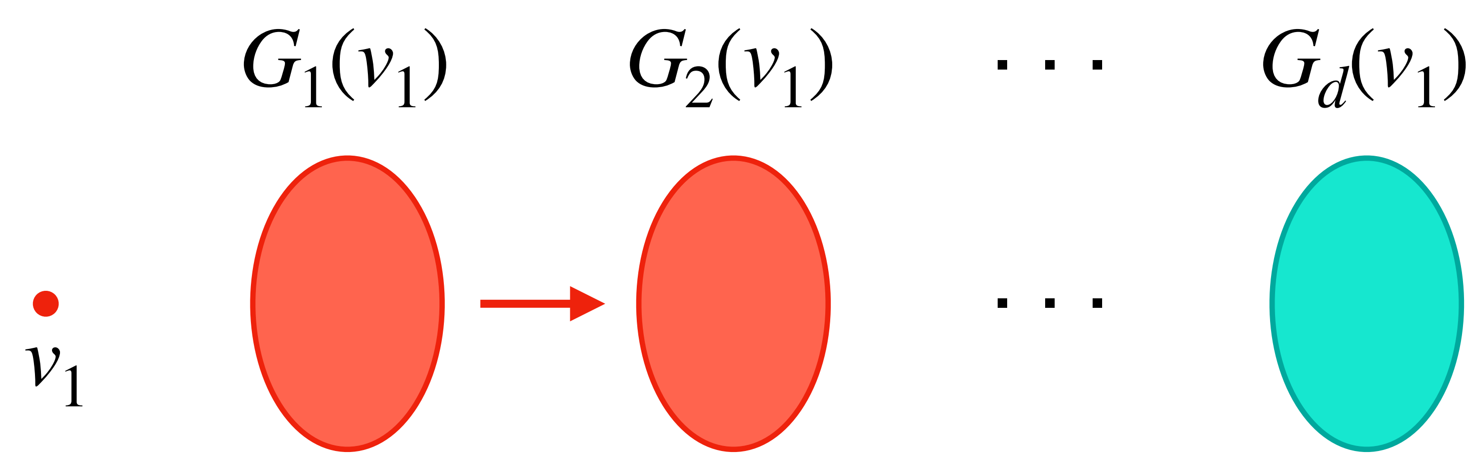


The burning number $b(G)$

- ∂ : the path-length distance on V
- $G_i(v) = \{u \in V : \partial(u, v) = i\} \quad (v \in V,\ i = 0,1,2,\ldots)$
- $d = \mathrm{diam}(G)$: the diameter of G

Lemma. $b(G) \leqslant d + 1.$

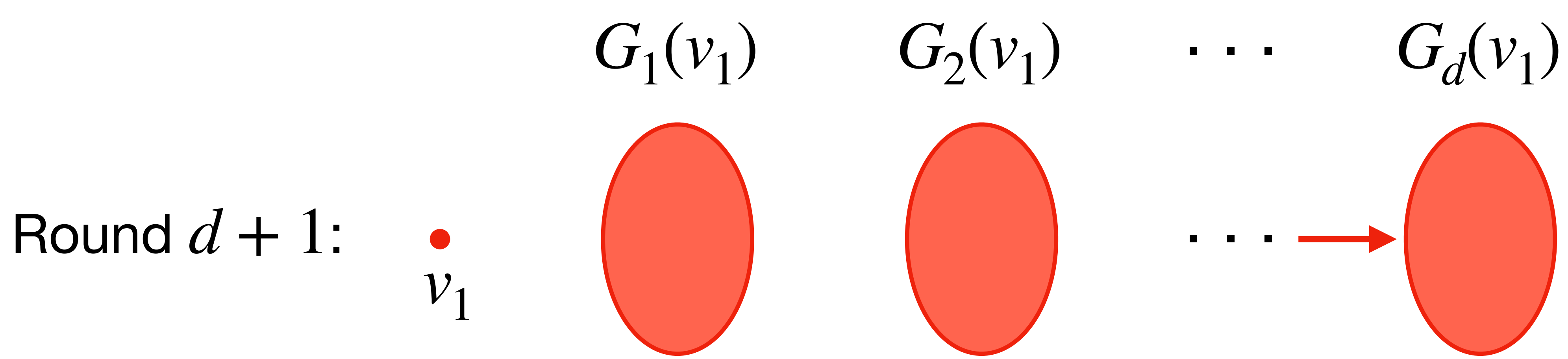
Round 3:



The burning number $b(G)$

- ∂ : the path-length distance on V
- $G_i(v) = \{u \in V : \partial(u, v) = i\} \quad (v \in V,\ i = 0,1,2,\ldots)$
- $d = \mathrm{diam}(G)$: the diameter of G

Lemma. $b(G) \leqslant d + 1$.



The burning number $b(G)$

- ∂ : the path-length distance on V
- $G_i(v) = \{u \in V : \partial(u, v) = i\} \quad (v \in V,\ i = 0,1,2,\ldots)$
- $d = \mathrm{diam}(G)$: the diameter of G

Lemma. $b(G) \leqslant d + 1.$

Burning Number Conjecture (Bonato et al., 2016)**.**

- $b(G) \leqslant \lceil n^{1/2} \rceil$, where $n = \#V$.

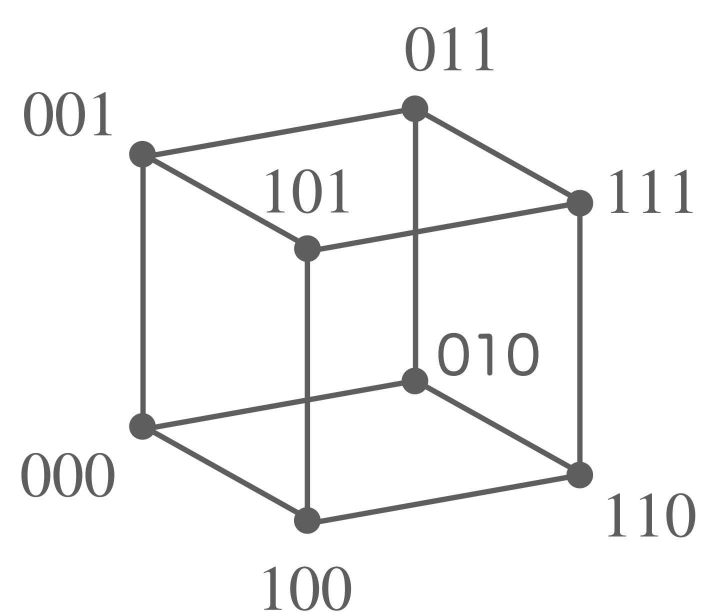
Example. The BNC holds for P_n and C_n.

Example. The BNC holds whenever G has a Hamiltonian path.

The burning number of the hypercubes

- Alon (1992) determined the burning number of the hypercubes.
- $Q_d = (V, E)$: the $\boldsymbol{d}$**-dimensional hypercube**
- $V = \{0,1\}^d$
- $(\epsilon_1, \epsilon_2, \ldots, \epsilon_d) \sim (\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_d) \overset{\text{def}}{\iff} \#\{i : \epsilon_i \neq \epsilon'_i\} = 1$

Q_3

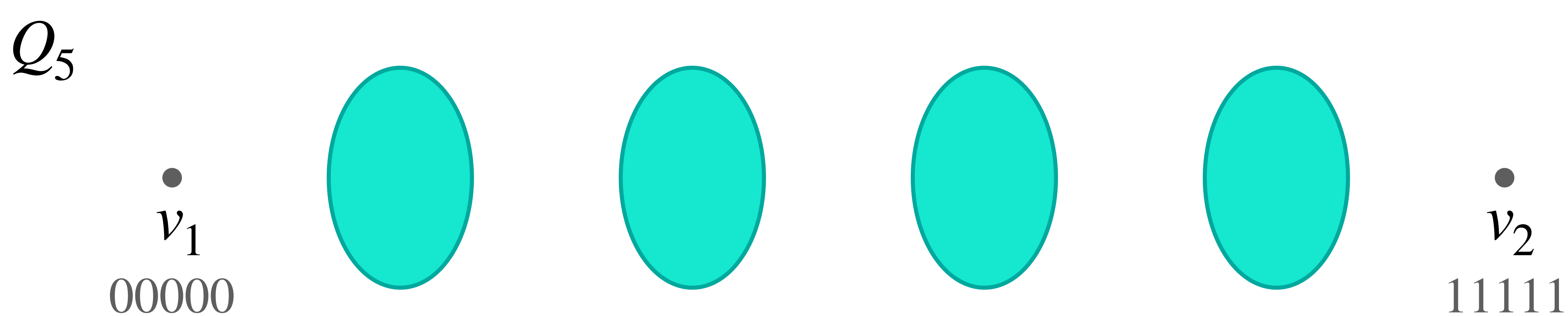


The burning number of the hypercubes

- Alon (1992) determined the burning number of the hypercubes.
- $Q_d = (V, E)$: the $\boldsymbol{d}$**-dimensional hypercube**
- $V = \{0,1\}^d$
- $(\epsilon_1, \epsilon_2, \ldots, \epsilon_d) \sim (\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_d) \overset{\text{def}}{\iff} \#\{i : \epsilon_i \neq \epsilon'_i\} = 1$

Theorem (Alon, 1992). $b(Q_d) = \lceil d/2 \rceil + 1.$

- $b(Q_d) \leqslant \lceil d/2 \rceil + 1$: Choose antipodal vertices v_1 and v_2.



The burning number of the hypercubes

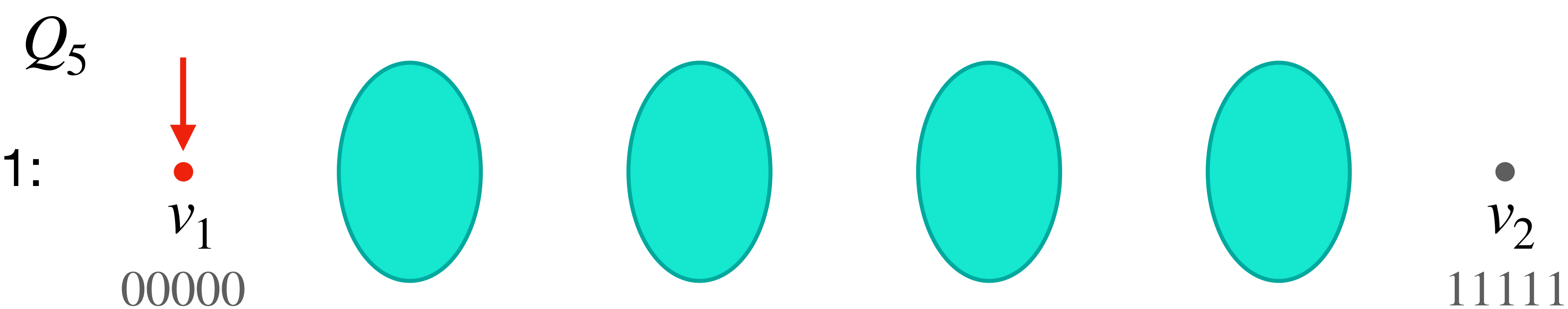
- Alon (1992) determined the burning number of the hypercubes.

- $Q_d = (V, E)$: the ***d*-dimensional hypercube**
- $V = \{0,1\}^d$
- $(\epsilon_1, \epsilon_2, \ldots, \epsilon_d) \sim (\epsilon_1', \epsilon_2', \ldots, \epsilon_d') \overset{\text{def}}{\Longleftrightarrow} \#\{i : \epsilon_i \neq \epsilon_i'\} = 1$

Theorem (Alon, 1992). $b(Q_d) = \lceil d/2 \rceil + 1.$

- $b(Q_d) \leqslant \lceil d/2 \rceil + 1$: Choose antipodal vertices v_1 and v_2.

Round 1:

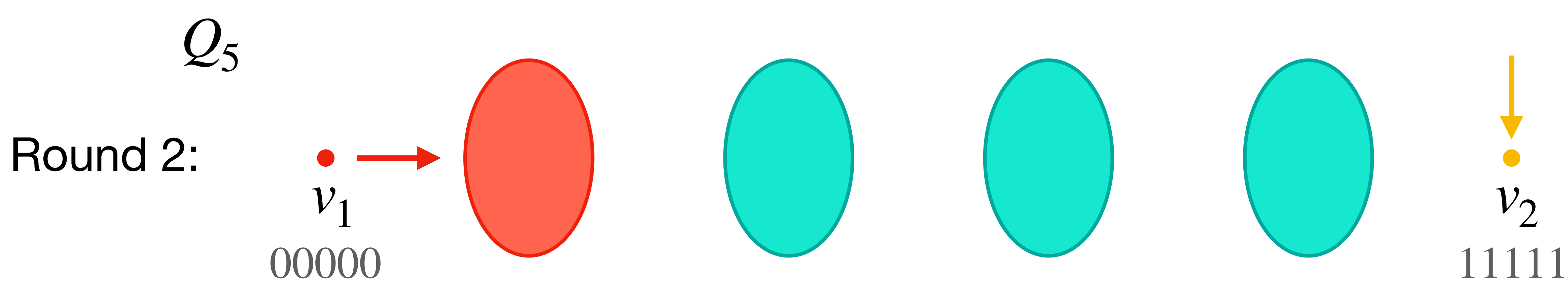


The burning number of the hypercubes

- Alon (1992) determined the burning number of the hypercubes.
- $Q_d = (V, E)$: the $\boldsymbol{d}$**-dimensional hypercube**
- $V = \{0,1\}^d$
- $(\epsilon_1, \epsilon_2, \ldots, \epsilon_d) \sim (\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_d) \overset{\text{def}}{\iff} \#\{i : \epsilon_i \neq \epsilon'_i\} = 1$

Theorem (Alon, 1992). $b(Q_d) = \lceil d/2 \rceil + 1.$

- $b(Q_d) \leqslant \lceil d/2 \rceil + 1$: Choose antipodal vertices v_1 and v_2.



The burning number of the hypercubes

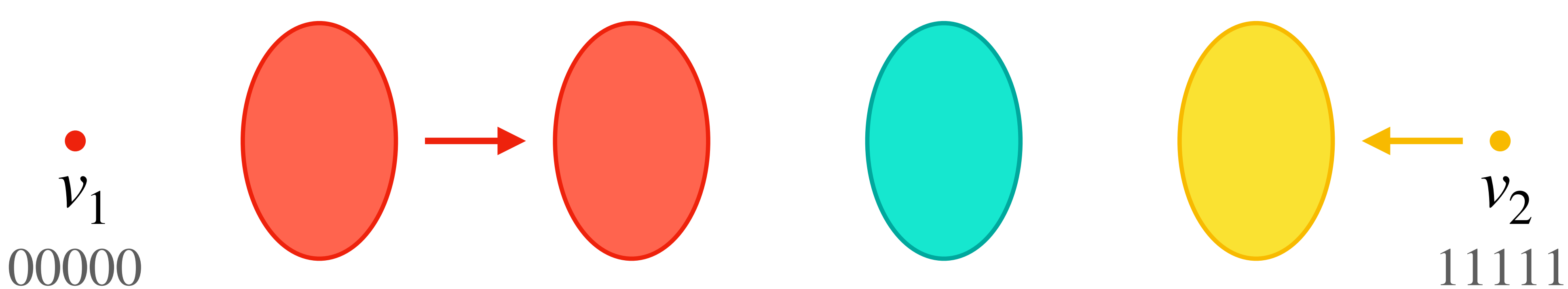
- Alon (1992) determined the burning number of the hypercubes.
- $Q_d = (V, E)$: the $\boldsymbol{d}$**-dimensional hypercube**
- $V = \{0,1\}^d$
- $(\epsilon_1, \epsilon_2, \ldots, \epsilon_d) \sim (\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_d) \overset{\text{def}}{\iff} \#\{i : \epsilon_i \neq \epsilon'_i\} = 1$

Theorem (Alon, 1992). $b(Q_d) = \lceil d/2 \rceil + 1.$

- $b(Q_d) \leqslant \lceil d/2 \rceil + 1$: Choose antipodal vertices v_1 and v_2.

Q_5

Round 3:



The burning number of the hypercubes

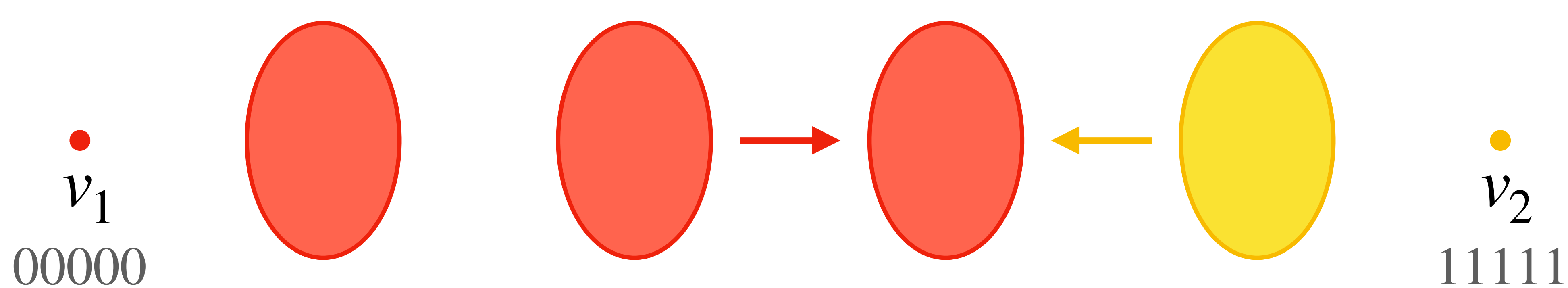
- Alon (1992) determined the burning number of the hypercubes.
- $Q_d = (V, E)$: the ***d*-dimensional hypercube**
- $V = \{0,1\}^d$
- $(\epsilon_1, \epsilon_2, \ldots, \epsilon_d) \sim (\epsilon_1', \epsilon_2', \ldots, \epsilon_d') \overset{\text{def}}{\iff} \#\{i : \epsilon_i \neq \epsilon_i'\} = 1$

Theorem (Alon, 1992). $b(Q_d) = \lceil d/2 \rceil + 1.$

- $b(Q_d) \leqslant \lceil d/2 \rceil + 1$: Choose antipodal vertices v_1 and v_2.

Q_5

Round 4:



The burning number of the hypercubes

- Alon (1992) determined the burning number of the hypercubes.

- $Q_d = (V, E)$: the **d-dimensional hypercube**
- $V = \{0,1\}^d$
- $(\epsilon_1, \epsilon_2, \ldots, \epsilon_d) \sim (\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_d) \overset{\text{def}}{\iff} \#\{i : \epsilon_i \neq \epsilon'_i\} = 1$

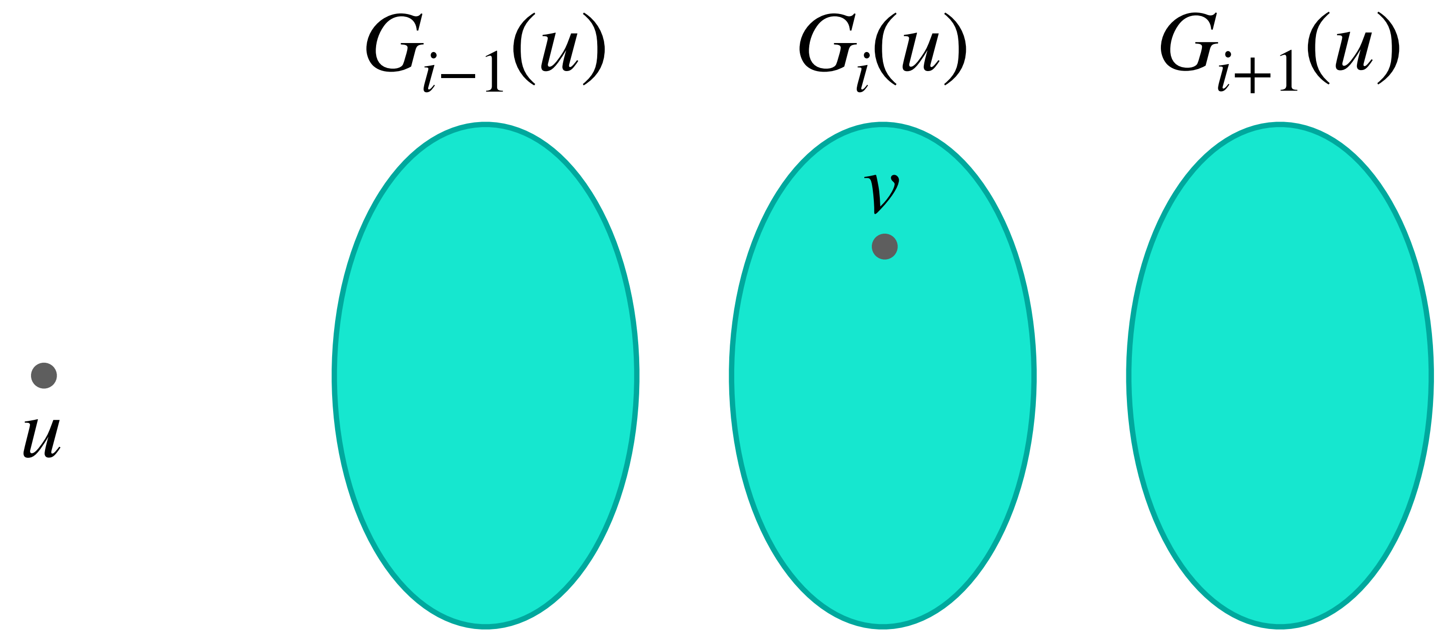
Theorem (Alon, 1992)**.** $b(Q_d) = \lceil d/2 \rceil + 1.$

- $b(Q_d) \leqslant \lceil d/2 \rceil + 1$: Choose antipodal vertices v_1 and v_2.
- $b(Q_d) \geqslant \lceil d/2 \rceil + 1$: Alon used a geometric method.

Beck-Fiala (1981)

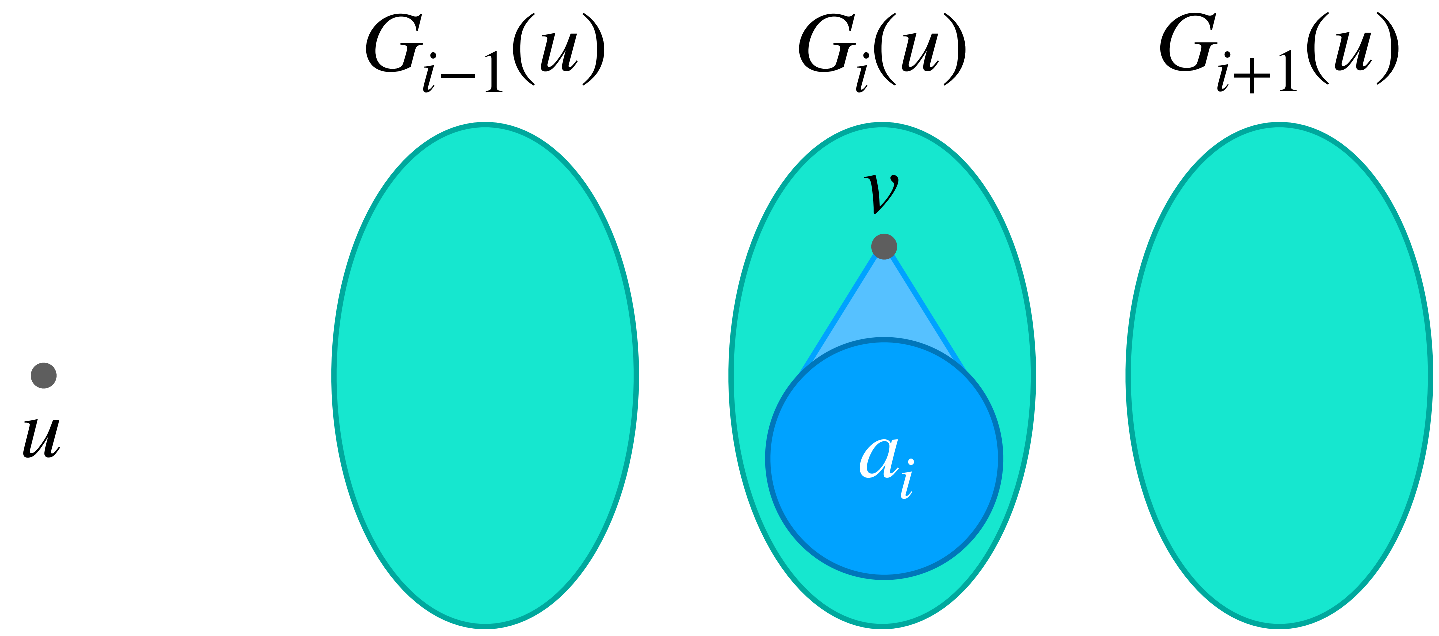
Distance-regular graphs

We assume G is **distance-regular** with diameter d.



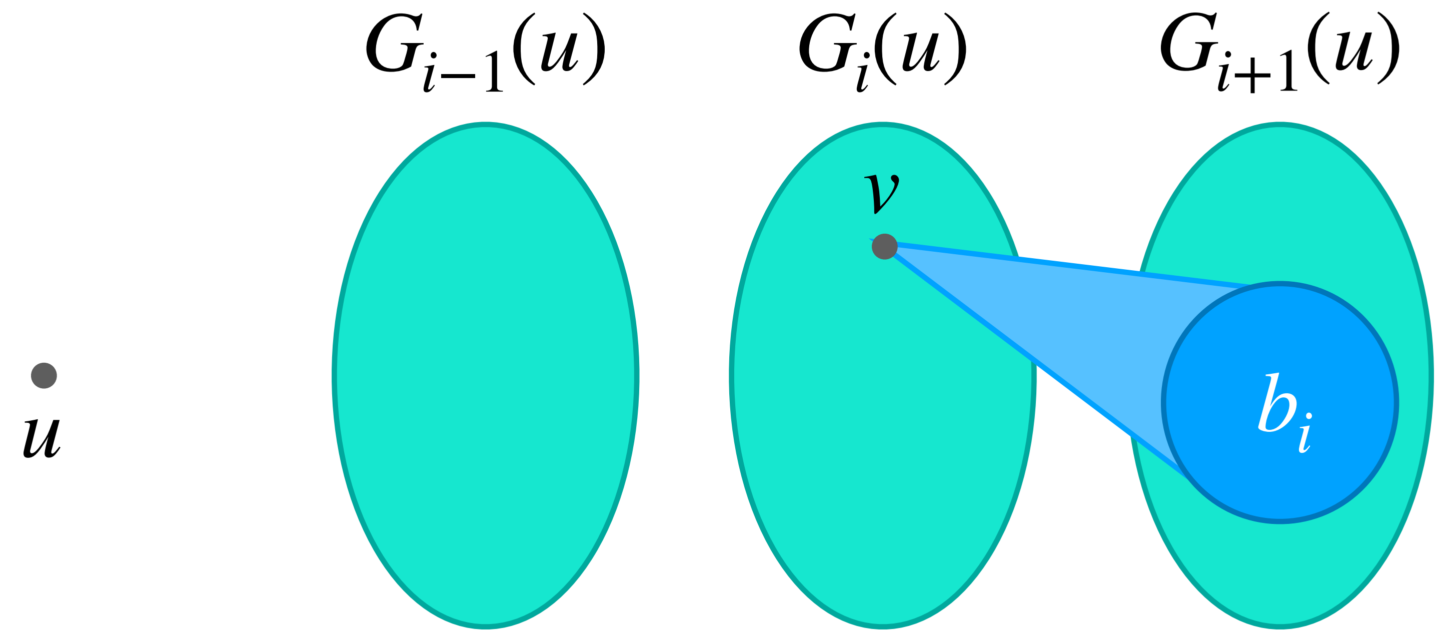
Distance-regular graphs

We assume G is **distance-regular** with diameter d.



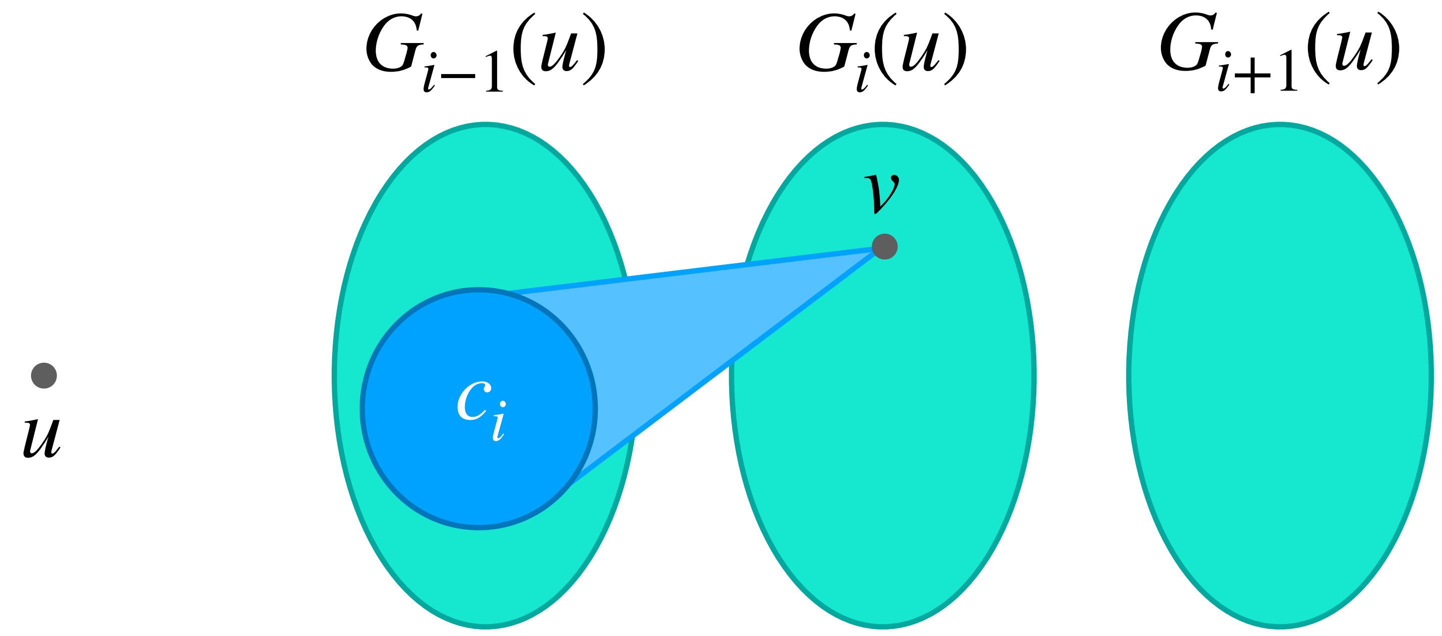
Distance-regular graphs

We assume G is **distance-regular** with diameter d.



Distance-regular graphs

We assume G is **distance-regular** with diameter d.



Examples.
- the hypercubes Q_d
- the cycles C_n

Remark. G is regular with valency b_0.

Eigenpolytopes of DRGs (Godsil, 1998)

- $A \in \mathbb{R}^{n \times n}$: the **adjacency matrix** of G

$$A_{uv} = \begin{cases} 1 & \text{if } u \sim v \\ 0 & \text{otherwise} \end{cases} \qquad (u, v \in V)$$

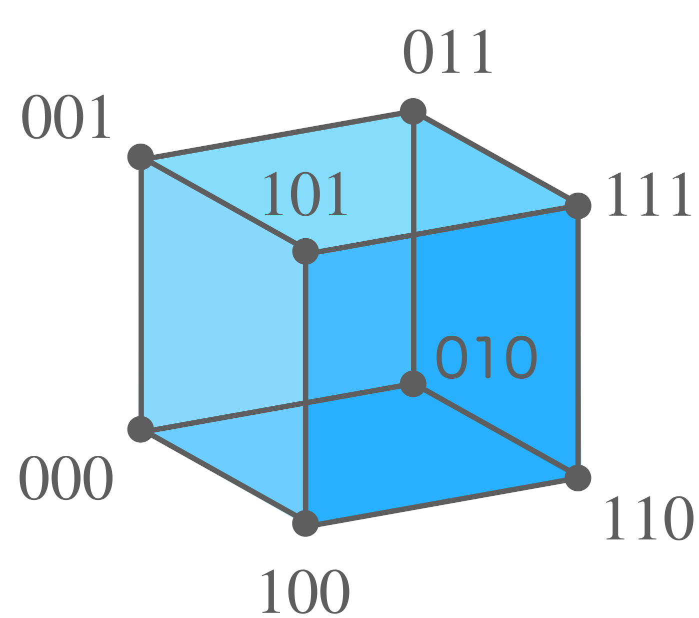
- $\theta_0 (= b_0) > \theta_1 > \cdots > \theta_d$: the distinct eigenvalues of A

multiplicity 1

- $E = E_{\theta_1}$: the orthogonal projection onto the eigenspace for θ_1
- $\mathrm{conv}\{Ev : v \in V\} \subset E\mathbb{R}^n \cong \mathbb{R}^m$: the **eigenpolytope** of G

v^{th} column of E multiplicity of θ_1

Q_3



$m = 3$

Eigenpolytopes of DRGs (Godsil, 1998)

- $A \in \mathbb{R}^{n\times n}$: the **adjacency matrix** of G

$$A_{uv} = \begin{cases} 1 & \text{if } u \sim v \\ 0 & \text{otherwise} \end{cases} \qquad (u, v \in V)$$

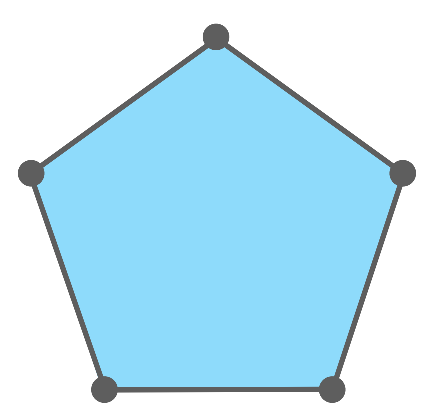
- $\theta_0 (= b_0) > \theta_1 > \cdots > \theta_d$: the distinct eigenvalues of A

  multiplicity 1

- $E = E_{\theta_1}$: the orthogonal projection onto the eigenspace for θ_1
- $\mathrm{conv}\{Ev : v \in V\} \subset E\mathbb{R}^n \cong \mathbb{R}^m$: the **eigenpolytope** of G

  v^{th} column of E multiplicity of θ_1

C_5



$m = 2$

Eigenpolytopes of DRGs (Godsil, 1998)

- $A \in \mathbb{R}^{n \times n}$: the **adjacency matrix** of G

$$A_{uv} = \begin{cases} 1 & \text{if } u \sim v \\ 0 & \text{otherwise} \end{cases} \qquad (u, v \in V)$$

- $\theta_0 (= b_0) > \theta_1 > \cdots > \theta_d$: the distinct eigenvalues of A

multiplicity 1

- $E = E_{\theta_1}$: the orthogonal projection onto the eigenspace for θ_1
- $\mathrm{conv}\{Ev : v \in V\} \subset E\mathbb{R}^n \cong \mathbb{R}^m$: the **eigenpolytope** of G

v^{th} column of E multiplicity of θ_1

Remark. $\|Ev\|^2 = m/n \ \ (v \in V)$.

The 1-skeleton of the eigenpolytope

Theorem (Godsil, 1998). G is the 1-skeleton of its eigenpolytope if and only if it is one of the following:

$\begin{cases} V = 0\text{-dim. facets} \\ E = 1\text{-dim. facets} \end{cases}$

(a) the Hamming graph $H(d, q)$ $(d \geqslant 1, q \geqslant 2)$

(b) the Johnson graph $J(s, d)$ $(s \geqslant 2d \geqslant 2)$

(c) the halved e-cube $(e \geqslant 2)$ $\longleftarrow$ $d = \lfloor e/2 \rfloor$

(d) the Schläfli graph $\longleftarrow$ $d = 2, b(G) = 3$

(e) the Gosset graph $\longleftarrow$ $d = 3, b(G) = 3$

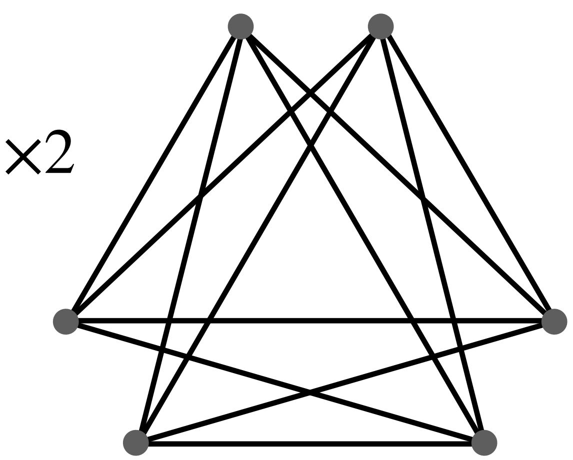
(f) the icosahedron $\longleftarrow$ $d = 3, b(G) = 3$

(g) the dodecahedron $\longleftarrow$ $d = 4, b(G) = 4$

(h) the complete multipartite graph $K_{r\times 2}$ $(r \geqslant 2)$ $\longleftarrow$ $d = 2$, $b(G) = 2$

(i) the n-cycle C_n $(n \geqslant 3)$ $\longleftarrow$ $d = \lfloor n/2 \rfloor$

$K_{3\times 2}$



The 1-skeleton of the eigenpolytope

Theorem (Godsil, 1998). G is the 1-skeleton of its eigenpolytope if and only if it is one of the following:

$\begin{cases} V = 0\text{-dim. facets} \\ E = 1\text{-dim. facets} \end{cases}$

(a) the Hamming graph $H(d, q)$ $(d \geqslant 1, q \geqslant 2)$

(b) the Johnson graph $J(s, d)$ $(s \geqslant 2d \geqslant 2)$

(c) the halved e-cube $(e \geqslant 2)$ $\longleftarrow$ $d = \lfloor e/2 \rfloor$

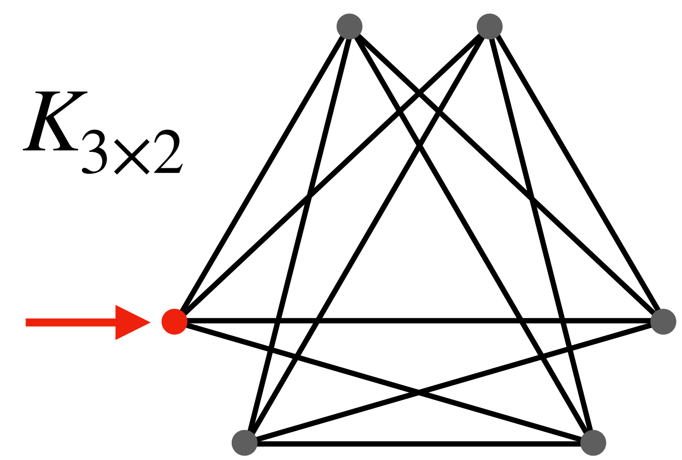
(d) the Schläfli graph $\longleftarrow$ $d = 2, b(G) = 3$

(e) the Gosset graph $\longleftarrow$ $d = 3, b(G) = 3$

(f) the icosahedron $\longleftarrow$ $d = 3, b(G) = 3$

(g) the dodecahedron $\longleftarrow$ $d = 4, b(G) = 4$

$K_{3\times 2}$



(h) the complete multipartite graph $K_{r\times 2}$ $(r \geqslant 2)$ $\longleftarrow$ $d = 2$, $b(G) = 2$

(i) the n-cycle C_n $(n \geqslant 3)$ $\longleftarrow$ $d = \lfloor n/2 \rfloor$

The 1-skeleton of the eigenpolytope

Theorem (Godsil, 1998). G is the 1-skeleton of its eigenpolytope if and only if it is one of the following:

$\begin{cases} V = 0\text{-dim. facets} \\ E = 1\text{-dim. facets} \end{cases}$

(a) the Hamming graph $H(d, q)$ $(d \geqslant 1, q \geqslant 2)$

(b) the Johnson graph $J(s, d)$ $(s \geqslant 2d \geqslant 2)$

(c) the halved e-cube $(e \geqslant 2)$ $\longleftarrow$ $d = \lfloor e/2 \rfloor$

(d) the Schläfli graph $\longleftarrow$ $d = 2, b(G) = 3$

(e) the Gosset graph $\longleftarrow$ $d = 3, b(G) = 3$

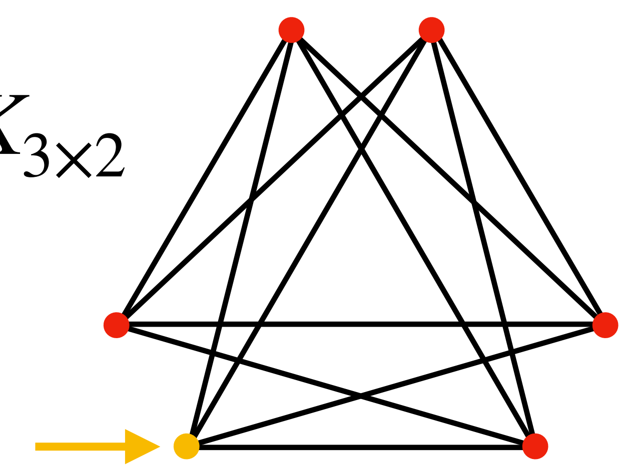
(f) the icosahedron $\longleftarrow$ $d = 3, b(G) = 3$

(g) the dodecahedron $\longleftarrow$ $d = 4, b(G) = 4$

(h) the complete multipartite graph $K_{r\times 2}$ $(r \geqslant 2)$ $\longleftarrow$ $d = 2$, $b(G) = 2$

(i) the n-cycle C_n $(n \geqslant 3)$ $\longleftarrow$ $d = \lfloor n/2 \rfloor$

$K_{3\times 2}$



The 1-skeleton of the eigenpolytope

Theorem (Godsil, 1998). G is the 1-skeleton of its eigenpolytope if and only if it is one of the following:

$$\begin{cases} V = 0\text{-dim. facets} \\ E = 1\text{-dim. facets} \end{cases}$$

(a) the Hamming graph $H(d, q)$ $(d \geqslant 1, q \geqslant 2)$

(b) the Johnson graph $J(s, d)$ $(s \geqslant 2d \geqslant 2)$

(c) the halved e-cube $(e \geqslant 2)$ ⟵ $d = \lfloor e/2 \rfloor$

(d) the Schläfli graph ⟵ $d = 2, b(G) = 3$

(e) the Gosset graph ⟵ $d = 3, b(G) = 3$

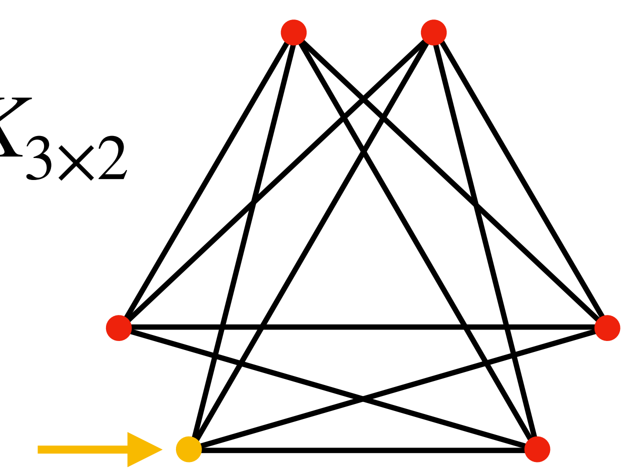
(f) the icosahedron ⟵ $d = 3, b(G) = 3$

(g) the dodecahedron ⟵ $d = 4, b(G) = 4$

(h) the complete multipartite graph $K_{r\times 2}$ $(r \geqslant 2)$ ⟵ $d = 2$, $b(G) = 2$

(i) the n-cycle C_n $(n \geqslant 3)$ ⟵ $d = \lfloor n/2 \rfloor$, $b(G) = \lceil n^{1/2} \rceil$

$K_{3\times 2}$



The Hamming and Johnson graphs

- The **Hamming graph** $H(d, q)$ has vertex set

$$V = \{0,1,\ldots,q-1\}^d,$$

where

$$(\epsilon_1, \epsilon_2, \ldots, \epsilon_d) \sim (\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_d) \overset{\text{def}}{\iff} \#\{i : \epsilon_i \neq \epsilon'_i\} = 1.$$

- The **Johnson graph** $J(s, d)$ has vertex set

the set of d-subsets

$$V = \binom{\{1,2,\ldots,s\}}{d},$$

where

$$u \sim v \overset{\text{def}}{\iff} \#(u \cap v) = d - 1.$$

The main result

Theorem (T.-Tokushige, 2025+)**.**

- For the Hamming graphs,

$$\left\lfloor \left(1-1/q\right)d \right\rfloor < b(H(d,q)) \leqslant \left\lfloor \left(1-1/q\right)d + (q+1)/2 \right\rfloor.$$

- For the Johnson graphs, ← The upper/lower bounds do not fit here!

$$b(J(s,d)) = d+1 \quad \text{for} \quad s > d^2,$$

$$b(J(2d,d)) = \lceil d/2 \rceil + 1.$$

- For the halved e-cubes,

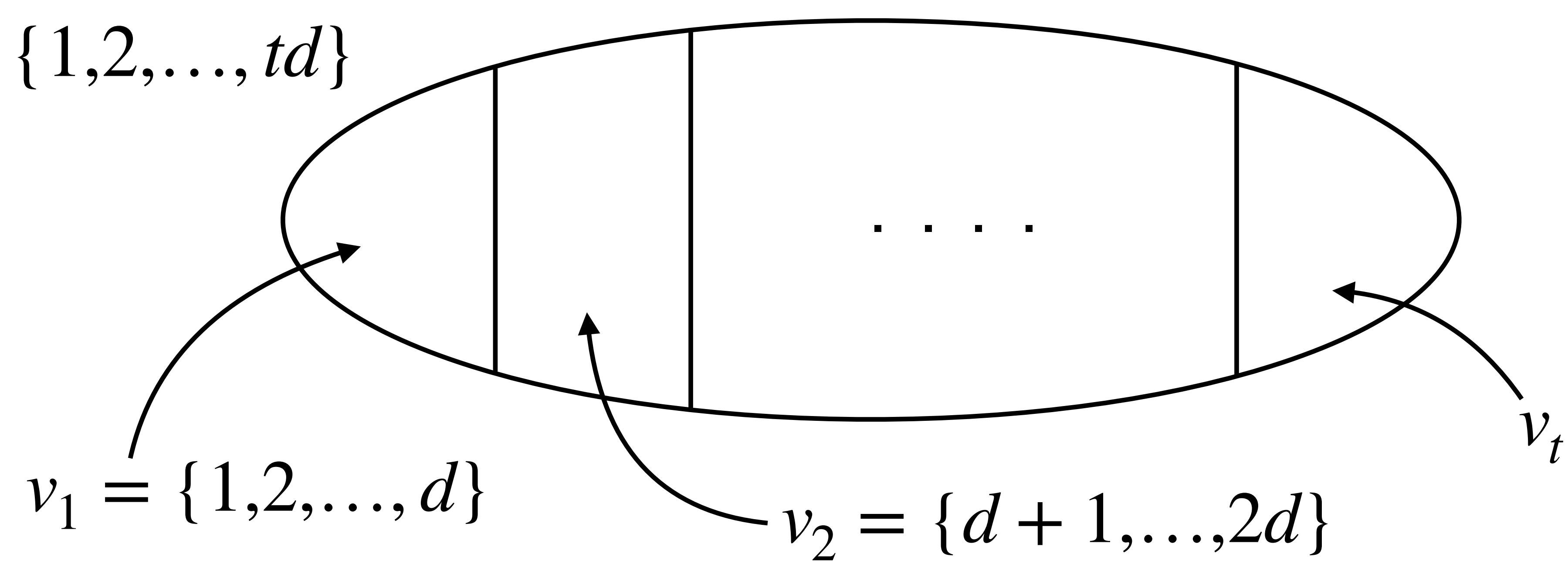
$$b\left(\tfrac{1}{2}Q_e\right) = \lceil e/4 \rceil + 1.$$

Proving the upper bounds

- For the Hamming graph $H(d, q)$, let

$$
\begin{aligned}
v_1 &= (0,0,\ldots,0),\\
v_2 &= (1,1,\ldots,1),\\
&\vdots\\
v_q &= (q-1,q-1,\ldots,q-1).
\end{aligned}
$$

- For the Johnson graph $J(td, d)$ $(t \geqslant 2)$ for example, let



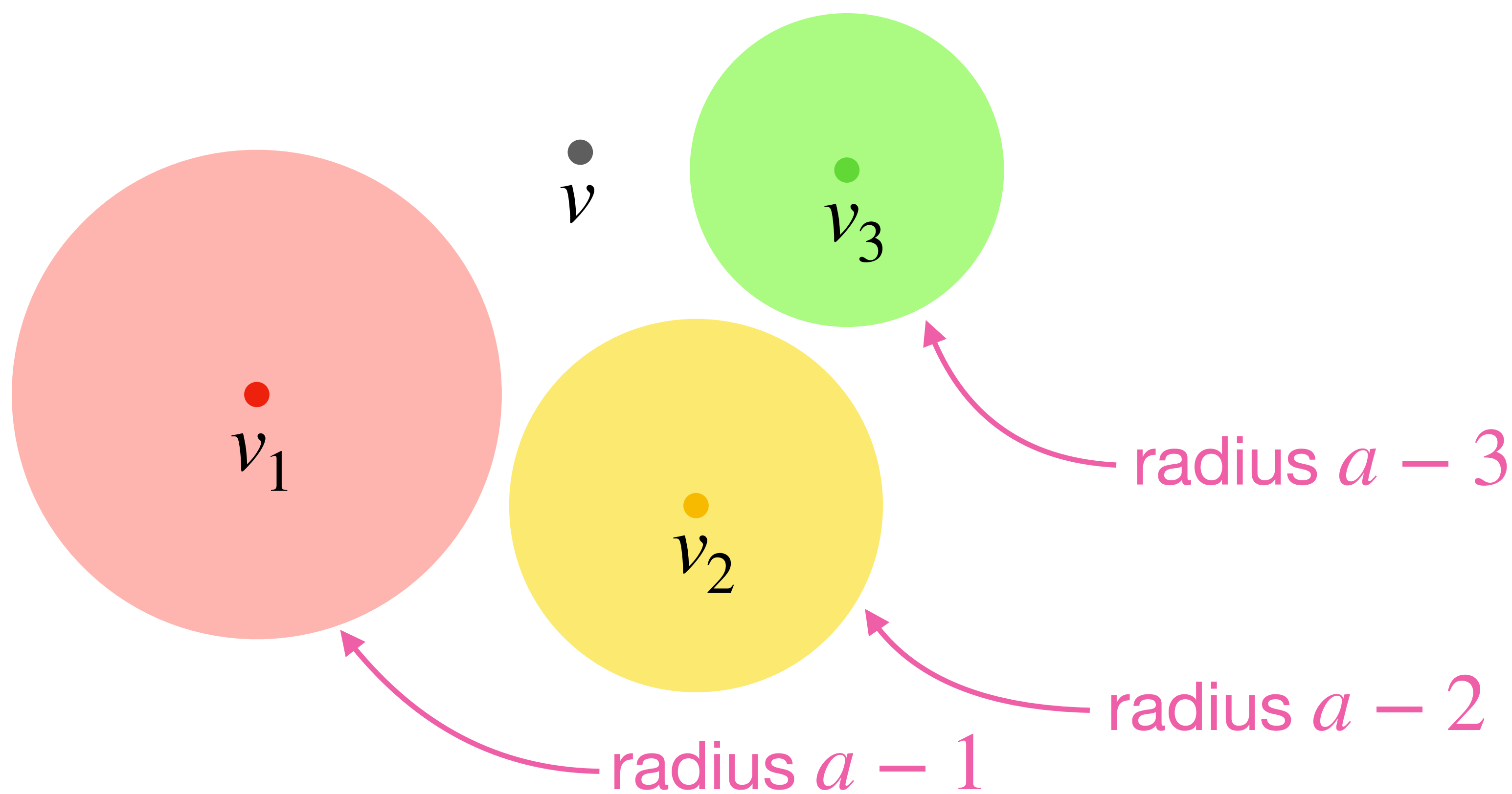
Proving the lower bounds

- $\lfloor (1-1/q)d \rfloor < b(H(d,q))$

- We consider $H(d,q)$. Recall it is the 1-skeleton of the eigenpolytope.
- Let $v_1, v_2, \ldots, v_a \in V$, where $a = \lfloor (1-1/q)\, d \rfloor$.
- We show $v_1, v_2, \ldots, v_a$ is **not** a burning sequence, i.e., for some $v \in V$ we have

$$\partial(v_i, v) > a - i \quad (i = 1,2,\ldots,a).$$

Round a:



Proving the lower bounds

- $a = \lfloor (1 - 1/q)d \rfloor$
- $\partial(v_i, v) > a - i$

- Pick any $v_{a+1}, v_{a+2}, \ldots, v_{m-1} \in V$.
- We construct pairs (x_i, F_i) $(i = 1,2,\ldots,m-1)$, where $x_i \in F_i$ and F_i is an i-dimensional facet.

Algorithm.

the origin in $\mathbb{R}^m$

- Set $x_m = 0$, and let F_m be the whole eigenpolytope.
- For $i = m, m-1, \ldots, 3, 2$, do the following.
  - ✻ Pick a line ℓ through x_i in $\operatorname{aff} F_i$ and in the solution space of $v_1 \bullet x = \cdots = v_{i-1} \bullet x = 0$.

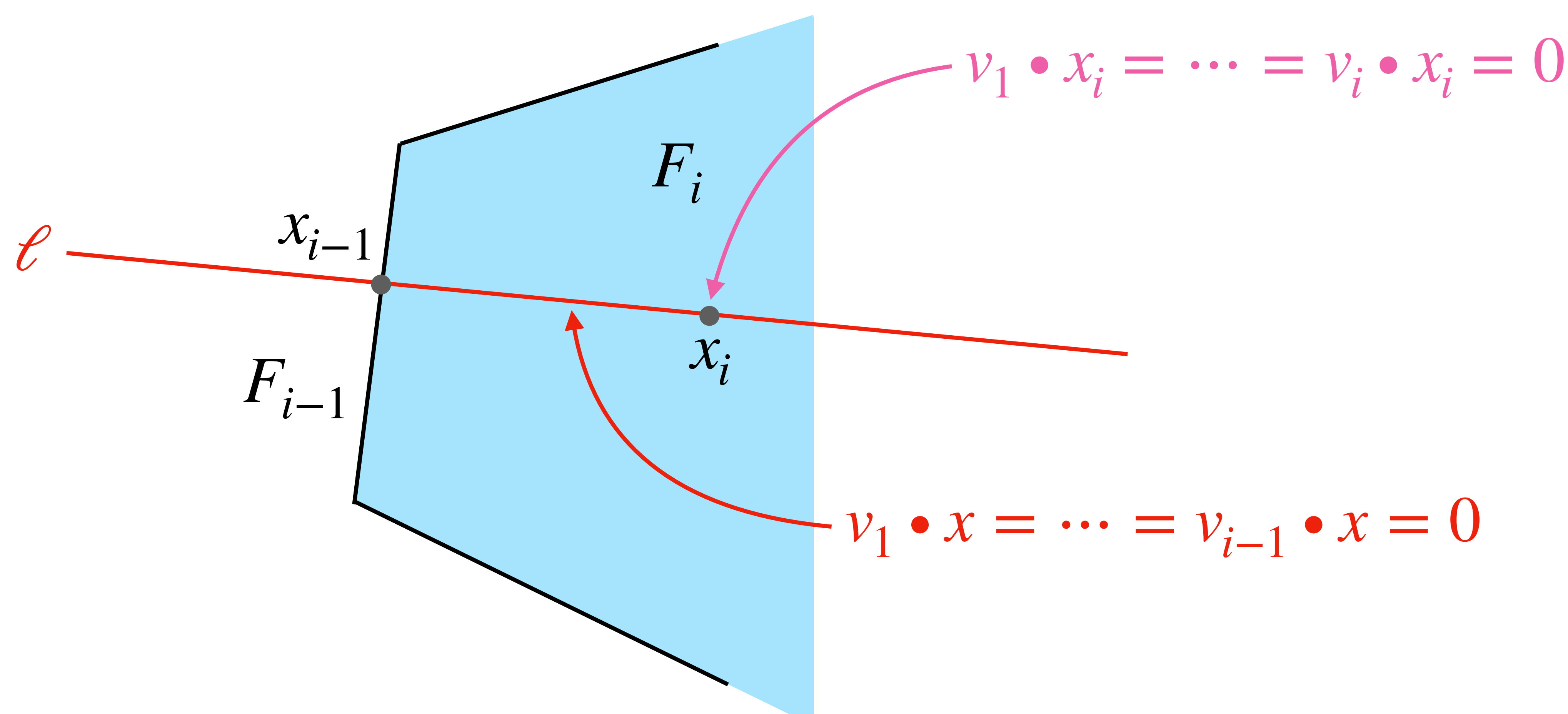
    the standard inner product
  - ✻ Let x_{i-1} be an endpoint of $\ell \cap F_i$, and let $F_{i-1} \subset F_i$ be an $(i-1)$-dimensional facet containing x_{i-1}.

- Finally, let v be an endpoint of F_1 s.t. $v_1 \bullet v \leqslant 0$.

Proving the lower bounds

- $a = \lfloor (1 - 1/q)d \rfloor$
- $\partial(v_i, v) > a - i$

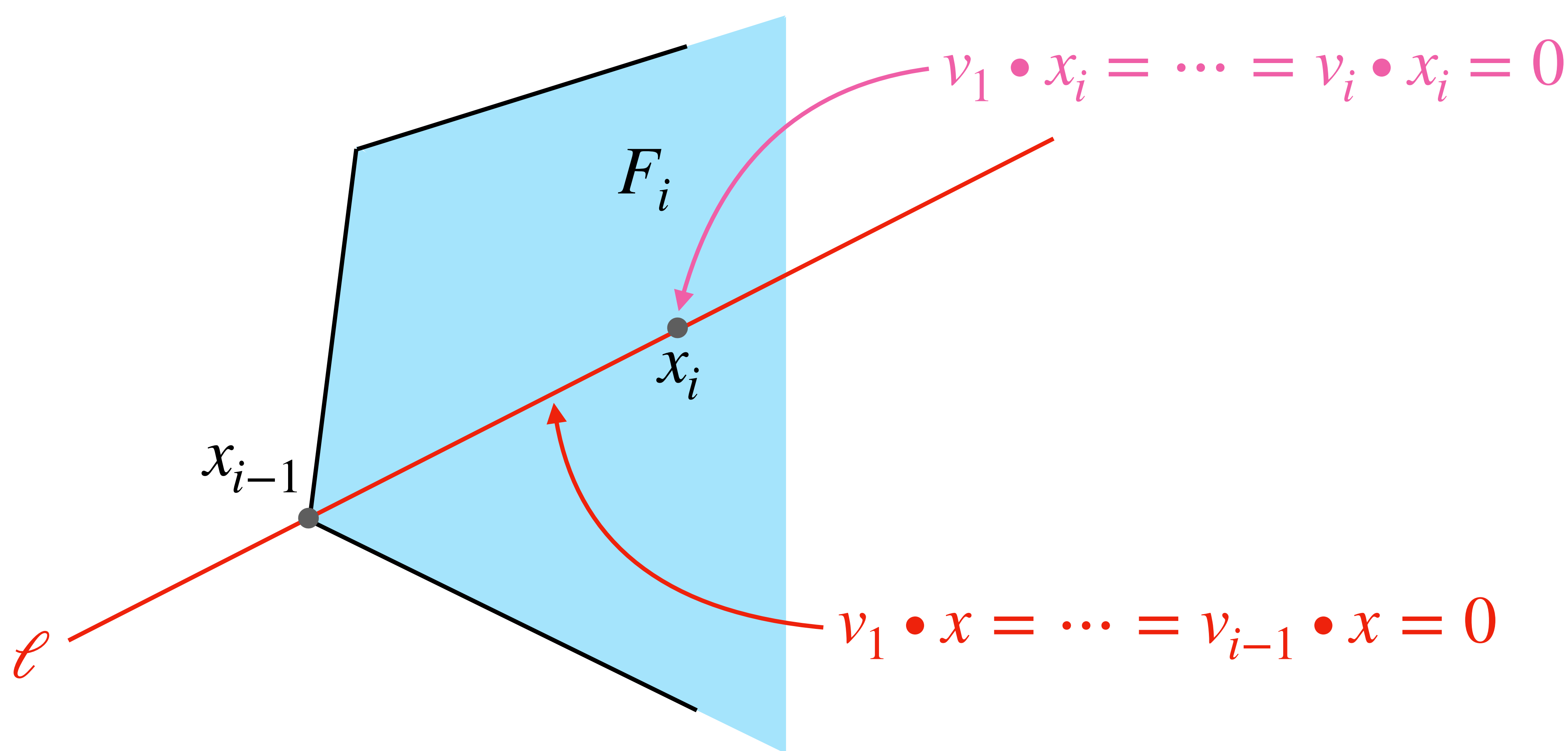
- Pick a line ℓ through x_i in $\operatorname{aff} F_i$ and in the solution space of $v_1 \bullet x = \cdots = v_{i-1} \bullet x = 0$.
- Let x_{i-1} be an endpoint of $\ell \cap F_i$, and let $F_{i-1} \subset F_i$ be an $(i-1)$-dimensional facet containing x_{i-1}.



Proving the lower bounds

- $a = \lfloor (1 - 1/q) d \rfloor$
- $\partial(v_i, v) > a - i$

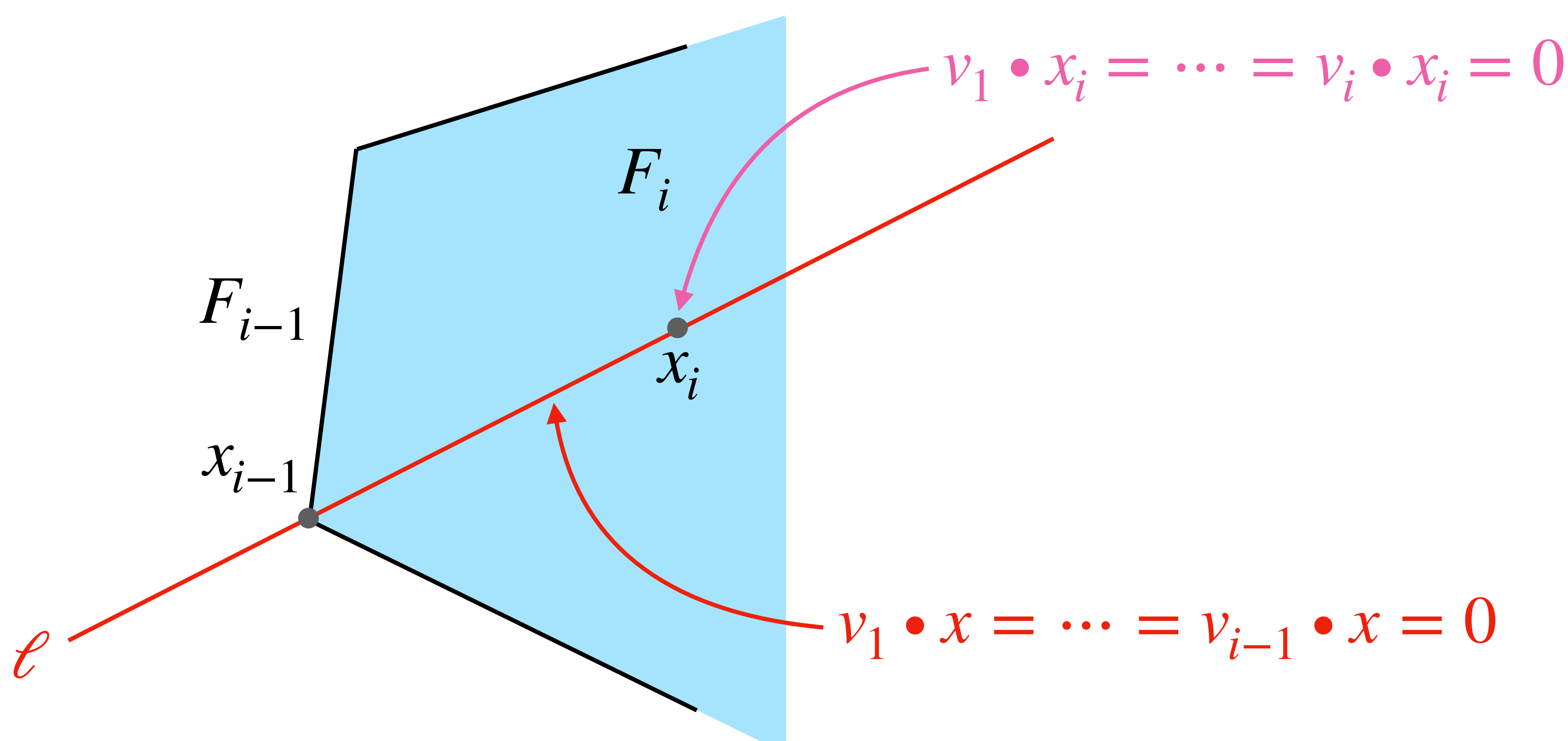
- Pick a line ℓ through x_i in $\operatorname{aff} F_i$ and in the solution space of $v_1 \bullet x = \cdots = v_{i-1} \bullet x = 0$.
- Let x_{i-1} be an endpoint of $\ell \cap F_i$, and let $F_{i-1} \subset F_i$ be an $(i-1)$-dimensional facet containing x_{i-1}.



Proving the lower bounds

- $a = \lfloor (1 - 1/q) d \rfloor$
- $\partial(v_i, v) > a - i$

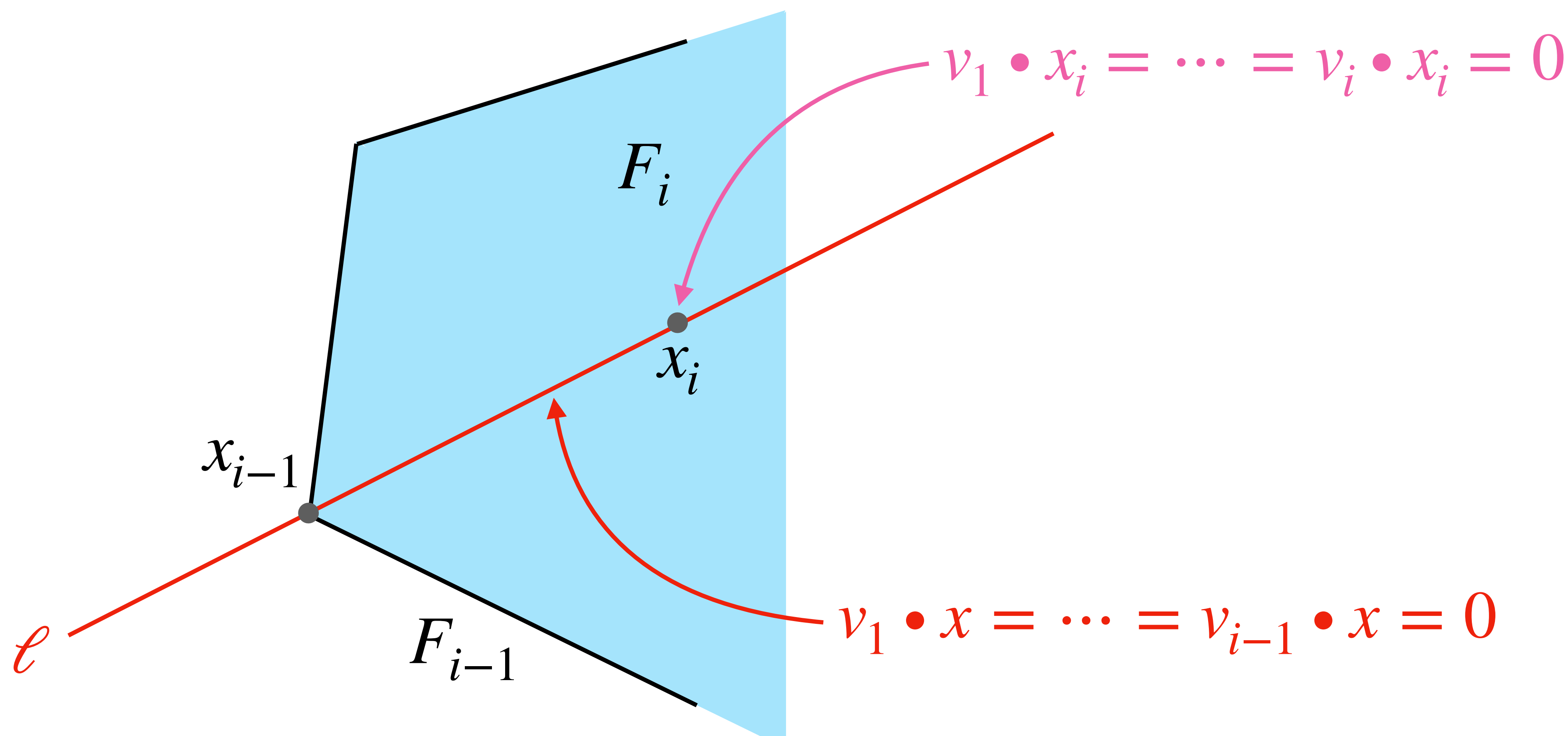
* Pick a line ℓ through x_i in $\operatorname{aff} F_i$ and in the solution space of $v_1 \bullet x = \cdots = v_{i-1} \bullet x = 0$.
* Let x_{i-1} be an endpoint of $\ell \cap F_i$, and let $F_{i-1} \subset F_i$ be an $(i-1)$-dimensional facet containing x_{i-1}.



Proving the lower bounds

- $a = \lfloor (1 - 1/q) d \rfloor$
- $\partial(v_i, v) > a - i$

* Pick a line ℓ through x_i in $\operatorname{aff} F_i$ and in the solution space of $v_1 \bullet x = \cdots = v_{i-1} \bullet x = 0$.
* Let x_{i-1} be an endpoint of $\ell \cap F_i$, and let $F_{i-1} \subset F_i$ be an $(i-1)$-dimensional facet containing x_{i-1}.

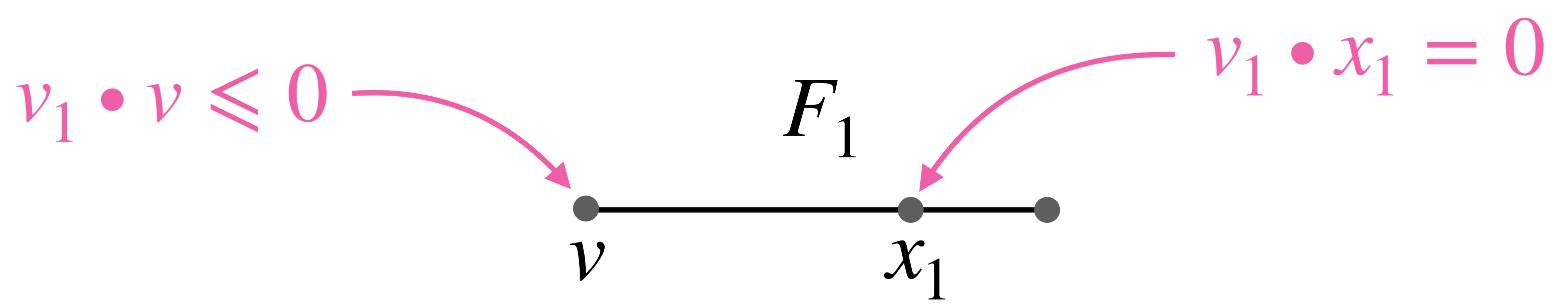


Proving the lower bounds

- $a = \lfloor (1 - 1/q) d \rfloor$
- $\partial(v_i, v) > a - i$

* Pick a line ℓ through x_i in $\operatorname{aff} F_i$ and in the solution space of $v_1 \bullet x = \cdots = v_{i-1} \bullet x = 0$.
* Let x_{i-1} be an endpoint of $\ell \cap F_i$, and let $F_{i-1} \subset F_i$ be an $(i-1)$-dimensional facet containing x_{i-1}.

- Finally, let v be an endpoint of F_1 s.t. $v_1 \bullet v \leqslant 0$.

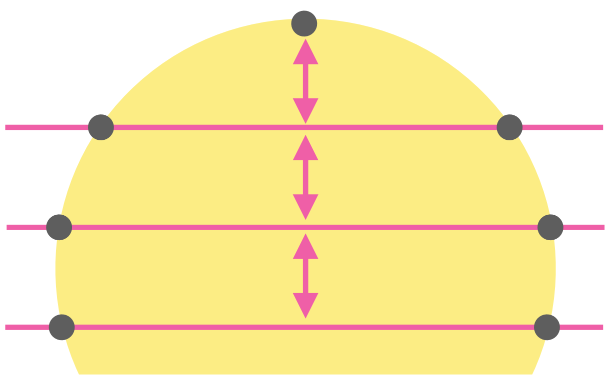


Proving the lower bounds

- $a = \lfloor (1-1/q)d \rfloor$
- $\partial(v_i, v) > a - i$

- Pick a line ℓ through x_i in $\operatorname{aff} F_i$ and in the solution space of $v_1 \bullet x = \cdots = v_{i-1} \bullet x = 0$.
- Let x_{i-1} be an endpoint of $\ell \cap F_i$, and let $F_{i-1} \subset F_i$ be an $(i-1)$-dimensional facet containing x_{i-1}.

- Finally, let v be an endpoint of F_1 s.t. $v_1 \bullet v \leqslant 0$.



Proposition. $\partial(v_i, v) > a - i \ (i = 1, 2, \ldots, a)$.

Ingredients.

(1) The facets correspond to convex subsets (Godsil, 1998).

(2) These graphs have **classical parameters** with base 1.

The main result

Theorem (T.-Tokushige, 2025+)**.**

- For the Hamming graphs,

$$\left\lfloor \left(1-1/q\right)d \right\rfloor < b(H(d,q)) \leqslant \left\lfloor \left(1-1/q\right)d + (q+1)/2 \right\rfloor.$$

- For the Johnson graphs, The upper/lower bounds do not fit here!

$$b(J(s,d)) = d+1 \text{ for } s > d^2,$$

$$b(J(2d,d)) = \lceil d/2 \rceil + 1.$$

- For the halved e-cubes,

$$b\left(\tfrac{1}{2}Q_e\right) = \lceil e/4 \rceil + 1.$$

Example. $2t < b(H(3t,3)) \leqslant 2t+2$**.**

Determine the exact value!!