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Master thesis

On Laplacian eigenvalues of trees

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G : a finite simple graph V = the vertices, E = the edges of G.

$L(G)$ = the Laplacian matrix of G $= (l_{uv})_{u,v \in V}$

$$l_{uu} = d(u) = \text{the degree at } u$$

$$\underset{u \neq v}{l_{uv}} = \begin{cases} -1 & uv \in E \\ 0 & uv \notin E \end{cases}$$

$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{|V|}$ the Laplacian eigenvalues of G.

$$\sigma(G) := \{\lambda_1, \ldots, \lambda_{|V|}\}$$

$$LE(G) = \sum_{i=1}^{|V|} |\lambda_i - \bar{d}|, \quad \text{where} \quad \bar{d} = \frac{1}{|V|}\sum_{u \in V} d(u) = \frac{2|E|}{|V|}$$

the Laplacian energy of G. the average degree of G

Examples

(1) P_n = the path with n vertices

$$\sigma(P_n) = \left\{ 2 + 2\cos\frac{n+1-i}{n}\pi \;\middle|\; i = 1, \cdots, n \right\}$$

$$LE(P_n) = 2 + 4\sum_{i=1}^{\lceil \frac{n}{2} \rceil} \cos\left(\frac{\pi}{n} i\right) + \frac{1}{n}\left((-1)^n - 1\right).$$

(2) S_n = the star with $n-1$ leaves (= pendants) $= K_{1,n-1}$

$$\sigma(S_n) = \{0, 1^{(n-2)}, n\}$$

$$LE(S_n) = 2\left(n - 2 + \frac{2}{n}\right).$$

Conjecture (Radenković - Gutman 2007 J. Serb. Chem. Soc 72).

T: a tree with n vertices

open

$$LE(P_n) \leqslant LE(T_n) \leqslant LE(S_n).$$

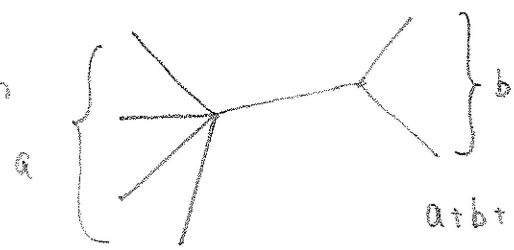
$$\uparrow$$

OK

Fritcher - Hoppen - Rocha - Trevisan 2011 Lin Alg Appl.

Trevisan - Carvalho - De Vecchio - Vinagre App. Math. Lett 24 (2011)

(1) OK for trees of the form



$a + b + 2 = n$.

(2) Some numerical experiments.

Conjecture 2 (TCdVV) open

$\#\{\lambda_i \mid \lambda_i \leqslant \bar{d}\} \geqslant \lceil \frac{n}{2} \rceil$ for each tree with n vertices

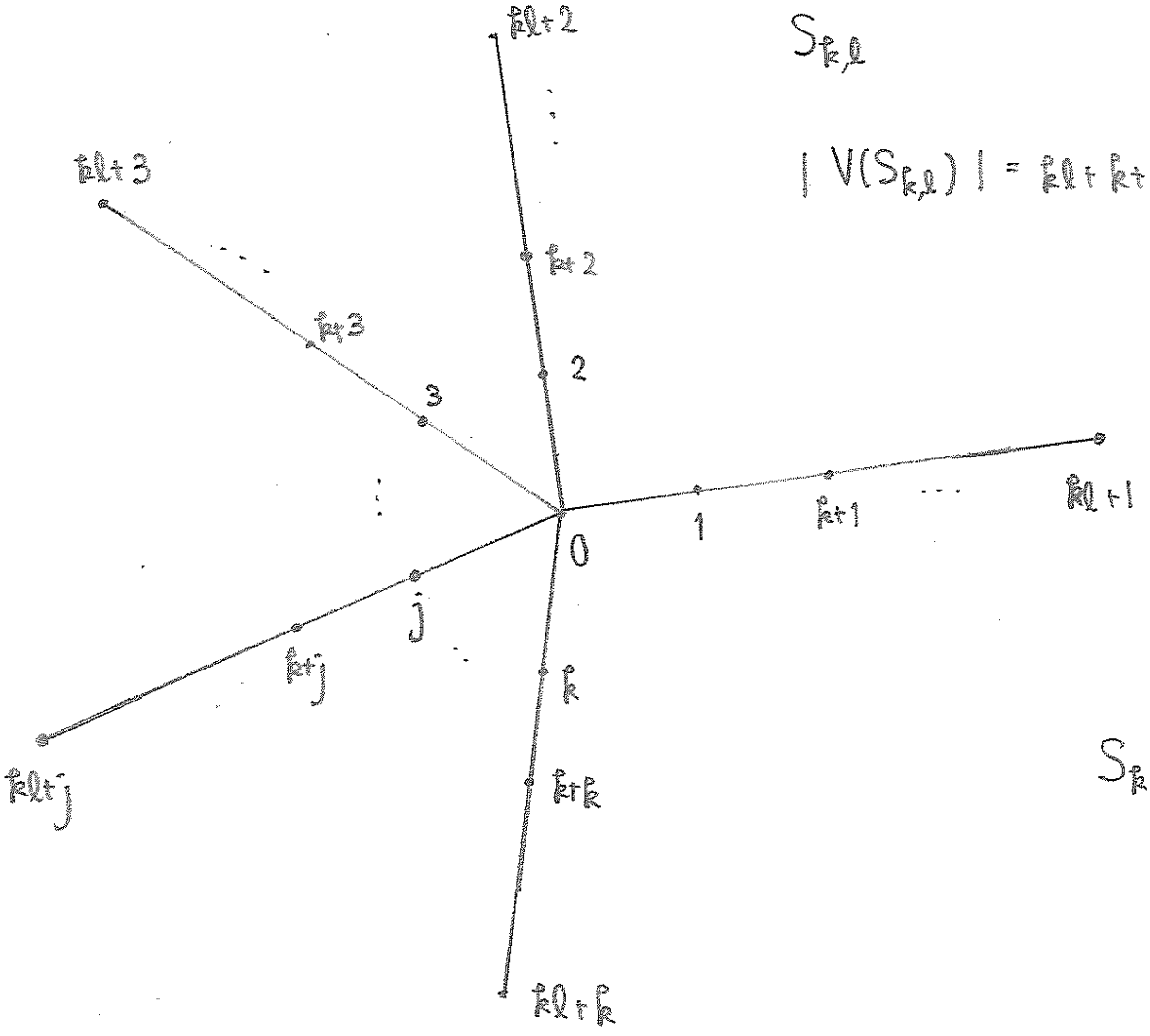
Braga - Rodrigues - Trevisan Dis. Math 313 (2013).

If Conjecture 2 holds for all trees T satisfying:

$u \in V(T) \quad d(u) \geqslant 3 \Rightarrow \#\{v \in N(u) \mid d(v) = 1\} \leqslant d(u) - 2$,

then it holds for every tree.

Munakata



$S_{k,l}$

$|V(S_{k,l})| = kl + k + 1$.

$S_{k,0} = S_{k+1}$
$= K_{1,k}$

Theorem 1 (Munakata)

$$\sigma(S_{k,1}) = \left\{ 0,\ \left(\frac{3-\sqrt{5}}{2}\right)^{(k-1)},\ \frac{k+3-\sqrt{(k-1)^2+4}}{2},\ \left(\frac{3+\sqrt{5}}{2}\right)^{(k-1)},\ \frac{k+3+\sqrt{(k-1)^2+4}}{2} \right\}$$

$$\bar{d} = 2 - \frac{2}{2k+1}$$

Conjecture 2 verified.

$$\frac{3-\sqrt{5}}{2} = 2 + 2\cos\frac{4\pi}{5}, \quad \frac{3+\sqrt{5}}{2} = 2+2\cos\frac{2\pi}{5}$$

$$\in \sigma(P_5)$$

$l \geqslant 2$ fixed.

$\sigma(S_{k,l})$ $\quad k \geqslant 2$

Numerical experiments.

$S_{2,2}$	0	μ_1	$\nu_{1,2}$	μ_2	$\nu_{2,2}$	μ_3	$\nu_{3,2}$	7
			$\wedge$		$\wedge$		$\wedge$	
$S_{3,2}$	0	$\mu_1^{(2)}$	$\nu_{1,3}$	$\mu_2^{(2)}$	$\nu_{2,3}$	$\mu_3^{(2)}$	$\nu_{3,3}$	10
			$\wedge$		$\wedge$		$\wedge$	
$S_{4,2}$	0	$\mu_1^{(3)}$	$\nu_{1,4}$	$\mu_1^{(3)}$	$\nu_{2,4}$	$\mu_3^{(3)}$	$\nu_{3,4}$	13

Conjecture

The above can be observed for $S_{k,l}$ $k \to \infty$, l : fixed.

Theorem 2 (Munakata) $\quad \ell \geq 1$ fixed.

Let $P_\ell(y), \cdots, P_0(y), P_{-1}(y)$ be polynomials defined by

$$P_\ell(y) = 1, \quad P_{\ell-1}(y) = 1 - y$$

$$P_i(y) = (2-y)\, P_{i+1}(y) - P_{i+2}(y) \qquad i = \ell-1, \cdots, -1.$$

Let $k \geq 2$.
Set $Q_\ell(y) = P_{-1}(y)$ and choose $a_1, \cdots, a_k$ such that $\sum_{i=1}^{k} a_i = 0$.

Take a root λ of $Q_\ell(\lambda) = 0$ and define $\qquad \deg Q_\ell = \ell + 1$.

$$\begin{cases} x_{ik+j} = P_i(\lambda)\, a_j & i = 0, \cdots, \ell, \ j = 1, \cdots, k \\ x_0 = 0 \end{cases}$$

Then

$$L(S_{k,\ell})\, x = \lambda x \qquad \text{for } x = (x_\nu)_{\nu = 0, \cdots, k\ell + k + 1}$$

In particular

$\lambda \in \sigma(S_{k,\ell}) \quad \forall k \geq 2$. and thus $\lambda \in \sigma(S_{2,\ell}) = \sigma(P_{2\ell+3})$
the multiplicity $\geq k-1$.

$$\sigma(P_{2\ell+3}) = \left\{ 2 + 2\cos \frac{2\ell+4-i}{2\ell+3} \;\middle|\; i = 1, \cdots, 2\ell+3 \right\}$$

Since $\deg Q_\ell = \ell+1$,

$$|\sigma(P_{2\ell+3}) \cap \sigma(S_{k,\ell})| \geq \ell+1 .$$

Question.

$$Q_\ell(\lambda) = 0 \Rightarrow \lambda = 2+2\cos\frac{2\ell+4-2i}{2\ell+3}\pi$$

$$= 2+2\cos\left(\frac{2(\ell+2-i)}{2\ell+3}\pi\right) \qquad i=1,\dots,\ell+1 \qquad ?$$

For convenience, we redefine

$R_i \longleftrightarrow P_{\ell-i-1}$

$\wr$

$R_\ell \longleftrightarrow P_{-1} = Q_\ell$

$$R_{-1}(y) = 1, \qquad R_0(y) = 1-y$$

$$R_{i+2}(y) = (2-y)R_{i+1}(y) - R_i(y). \qquad \text{and}$$

let

$$r_i(x) = R_i(2+2x).$$

Also let

$T_m(x)$ = the Chebyshev polynomial of the first kind defined by

$$T_m(\cos\theta) = \cos(m\theta).$$

Theorem

$$r_\ell(x) = (-1)^{\ell-1} \frac{T_{\ell+2}(x) - T_{\ell+1}(x)}{x-1}$$

Cor. $Q_\ell(\lambda) = 0$, $\lambda = 2+2\cos\theta$. $\theta = \frac{u\pi}{2\ell+3}$, $0 < \theta < \pi$

$$\Downarrow$$

$$r_\ell(\cos\theta) = R_\ell(2+2\cos\theta) = P_{-1}(2+2\cos\theta) = 0$$

$$\parallel$$

$$(-1)^{\ell-1} \frac{T_{\ell+2}(\cos\theta) - T_{\ell+1}(\cos\theta)}{\cos\theta - 1}$$

$$0 = T_{\ell+2}(\cos\theta) - T_{\ell+1}(\cos\theta) = \cos(\ell+2)\theta - \cos(\ell+1)\theta$$

$$= -2\sin\frac{2\ell+3}{2}\theta \sin\frac{\theta}{2}$$

Therefore

$$\theta = \frac{2\nu}{2\ell+3}\pi$$

Problem

Subdivision and spectra.